Effective action for superfluid Fermi systems in the strong-coupling limit

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(Received 21 December 2004; published 11 July 2005)

We derive the low-energy effective action for three-dimensional superfluid Fermi systems in the strong-coupling limit, where superfluidity originates from Bose-Einstein condensation of composite bosons. Taking into account density and pairing fluctuations on the same footing, we show that the effective action involves only the fermion density \( \rho_r \) and its conjugate variable, the phase \( \theta_r \) of the pairing order parameter \( \Delta_r \). We recover the standard action of a Bose superfluid of density \( \rho_r/2 \), where the bosons have a mass \( m_B = 2m \) and interact via a repulsive contact potential with amplitude \( g_B = 4\pi a_B/m_B \), \( a_B = 2a \) (\( a \) the \( s \)-wave scattering length associated to the fermion-fermion interaction in vacuum). For lattice models, the derivation of the effective action is based on the mapping of the attractive Hubbard model onto the Heisenberg model in a uniform magnetic field, and a coherent state path integral representation of the partition function. The effective description of the Fermi superfluid in the strong-coupling limit is a Bose-Hubbard model with an intersite hopping amplitude \( t_g = J/2 \) and an on-site repulsive interaction \( U_B = 2J_z \), where \( J = 4t^2/U \) (\( t \) and \( -U \) are the intersite hopping amplitude and the on-site attraction in the fermionic Hubbard model, \( z \) the number of nearest-neighbor sites). 

DOI: 10.1103/PhysRevA.72.013606

PACS number(s): 03.75.Hh, 74.20.Fg, 71.10.Fd, 05.30.Jp

I. INTRODUCTION

Recent progress in the experimental control of ultracold atomic Fermi [1–5] gases has revived the interest in the crossover from the weak-coupling BCS limit of superfluid fermions to the strong-coupling limit of condensed composite bosons [6,7]. In this paper, we derive the low-energy effective action for a superfluid Fermi system in the strong-coupling limit, both in continuum and lattice models. The latter may be relevant for high-\( T_c \) superconductors or ultracold Fermi gases in an optical lattice.

A Bose superfluid is described by a complex field \( \psi_r \) = \( \sqrt{\rho_r} e^{i\theta_r} \) where \( \rho_r \) is the boson density at position \( r \) in space. The equation of motion derived from the standard action of a Bose system leads to the Gross-Pitaevskii equation [8,9], i.e., a nonlinear Schrödinger equation for the \( \psi_r \) field. The Gross-Pitaevskii equation yields a simple description of quantum macroscopic phenomena like the Josephson effect or the flux quantization [10,11], and has proven to be a tool of choice for the understanding of many phenomena in ultracold atomic Bose gases [12]. In Fermi systems, there is, in general, no simple relation between the amplitude of the superfluid (pairing) order parameter \( \Delta_r \) and the fermion density \( \rho_r \). This suggests that a minimal description, aiming at making contact with the standard description of a Bose superfluid, should at least include the superfluid order parameter \( \Delta_r \) and the density \( \rho_r \) from the outset. In the strong-coupling limit, where superfluidity originates from Bose-Einstein condensation (BEC) of composite bosons, we expect the description in terms of \( \rho_r \) and \( \Delta_r \) = \( \sqrt{\rho_r} e^{i\theta_r} \) to be redundant and the superfluid to be described by a single complex field \( \phi_r = \sqrt{\rho_r/2} e^{i\theta_r} \) (\( \rho_r/2 \) being the density of composite bosons).

Previous studies of the BCS-BEC crossover in superfluid Fermi systems can be divided into two categories. In the first type of approach [6,13–18], the density \( \rho_r \) is not considered explicitly and a pairing field \( \Delta_r^{\text{HS}} \) is introduced by means of a Hubbard-Stratonovich transformation of the fermion-fermion interaction. In the BEC limit, the standard action \( S[\psi', \psi] \) of a Bose superfluid is recovered if one identifies \( \psi_r \) to \( \Delta_r^{\text{HS}} \) (after a proper rescaling). For a continuum model, the bosons have a mass \( m_B = 2m \) and interact via a repulsive contact potential with amplitude \( g_B = 4\pi a_B/m_B \), \( a_B = 2a \) (\( a \) the \( s \)-wave scattering length associated to the fermion-fermion interaction in vacuum). The main (conceptual) difficulty of this approach is that the Hubbard-Stratonovich field \( \Delta_r^{\text{HS}} \) is not the physical pairing field \( \Delta_r = \Delta_r^{\text{HS}} e^{i\theta_r} \) but rather its conjugate field [19]. Although both fields coincide at the mean-field level, they differ when fluctuations are taken into account. As a result, \( \psi_r \approx \Delta_r^{\text{HS}} \) does not correspond to \( \sqrt{\rho_r/2} e^{i\theta_r} \) as expected.

In the second type of approach to the BCS-BEC crossover [19–21], the physical density and pairing fields \( \rho_r \) and \( \Delta_r \) are introduced from the outset. For continuum models, only the weak-coupling limit has been considered [19,21]. For lattice (Hubbard) models in the strong-coupling low-density limit, one finds that the order parameter amplitude and the density are tied by the relation \( |\Delta_r| = \sqrt{\rho_r/2} \), so that the low-energy effective action can be written in terms of a single complex field \( \Delta_r = \sqrt{\rho_r/2} e^{i\theta_r} = \psi_r \) [19,20]. In the continuum limit, one finds that the (composite) bosons have a mass \( m_B = 1/J \) and interact via a repulsive contact potential with amplitude \( g_B = 8J \) (in two dimensions), where \( J = 4t^2/U \) (\( t \) being the intersite hopping amplitude and \( -U \) the on-site attractive interaction) [20].

Most of the theoretical works on the BCS-BEC crossover in ultracold atomic Fermi gases have been formulated within a fermion-boson model [22], aiming at incorporating the molecular states involved in the Feshbach resonance which drives the crossover. While the equivalence of the fermion-
The boson model to an effective single-channel model in the crossover region may be questionable [23,24], both models are equivalent in the strong-coupling limit.

The outline of the paper is as follows. In Sec. II, we extend the approach of Ref. [19] to the strong-coupling limit of a continuum model. The particle-particle and particle-hole channels are considered on the same footing, and the physical density (\(\rho_0\)) and pairing (\(\Delta_0\)) fields are introduced from the outset. The low-energy effective action is derived by assuming small fluctuations of the collective fields about their mean-field values. We find that fluctuations of \(\rho_0\) and \(\Delta_0\) are not independent, so that the low-energy action can be written in terms of a single complex field \(\psi_k = \sqrt{\rho_0} e^{i\chi_k}\). We recover the standard action of a Bose superfluid with \(m_B = 2m\) and \(g_B = 4\pi\alpha g/m_B\). For a lattice model (Sec. III), we follow the approach introduced in Ref. [20]. We map the attractive Hubbard model onto the half-filled repulsive Hubbard model in a uniform magnetic field coupled to the fermion spins. In the strong-coupling limit, the latter reduces to the Heisenberg model in a uniform field. The low-energy effective action of the attractive model is finally deduced from the Heisenberg model in a uniform field. The low-energy effective Hubbard model onto the half-filled repulsive Hubbard model is equivalent in the strong-coupling limit. The crossover region may be questionable.

The (real) density and (complex) pairing fields

\[
\rho_0 = c_r^\dagger c_r, \\
\Delta_0 = c_r^\dagger c_{r+},
\]

are equivalent in the strong-coupling limit. \(\rho_0\) and \(\Delta_0\) also determine the extension of the bound state. Low-energy properties depend solely on \(a\) (and not \(g\) or \(\Lambda\)); we shall therefore take the limit \(g \to 0\) and \(\Lambda \to \infty\) with a fixed \(a\). In the following, we consider the BEC limit defined by \(\rho_0 a^2 \ll 1 (a > 0)\), where superfluidity originates from BEC of composite bosons.

The (real) density and (complex) pairing fields

\[
\rho_0 = c_r^\dagger c_r, \\
\Delta_0 = c_r^\dagger c_{r+},
\]

can be introduced in the action by means of real \(\rho_0^{\text{HS}}, \rho_0^{\text{HS}}\) and complex \(\Delta_0^{\text{HS}}\) Lagrange multipliers:

\[
S = \int_0^\beta d\tau \int d^3r \left[ c_r^\dagger \left( \partial_\tau - \mu - \frac{\nabla^2}{2m} \right) c_r - g \psi_0 |\Delta_r|_0^2 - \frac{g\alpha}{4} (\rho_0^{\text{HS}} - \rho_0^{\text{HS}})^2 + i\rho_0^{\text{HS}} (\rho_0^{\text{HS}} - c_r^\dagger c_r) + i\rho_0^{\text{HS}} (\rho_0^{\text{HS}} - c_r^\dagger c_r) + c.c. \right],
\]

(3) denotes the Pauli matrices. Integrating over \(\rho_0^{\text{HS}}\), \(\rho_0^{\text{HS}}\), \(\Delta_0^{\text{HS}}\), and \(\rho_0, \Delta_0, \rho_0^{\text{HS}}, \Delta_0^{\text{HS}}\), we recover the original action (1) if we choose \(\alpha + \gamma = 1\). The relative weights \(\alpha\) and \(\gamma\) of the particle-hole and particle-particle channels are arbitrary. All the resulting effective actions are equivalent when treated exactly. However, to recover the mean-field results from a saddle-point approximation, we take \(\alpha = \gamma = 1\). When only low-energy long-wavelength fluctuations about the mean-field state are considered, there is no overlapping of the two channels and therefore no overcounting [19]. Note that by integrating out the physical fields \(\Delta_0, \rho_0\), one recovers the action \(S(\rho_0^{\text{HS}}, \rho_0^{\text{HS}}, \Delta_0^{\text{HS}})\) which is generally obtained by means of a Hubbard-Stratonovich decoupling of the interaction term. Thus the Lagrange multipliers \(\rho_0^{\text{HS}}, \rho_0^{\text{HS}}\), and \(\Delta_0^{\text{HS}}\) enforcing the constraints (3) can also be seen as Hubbard-Stratonovich fields [19]. In the following, we neglect spin fluctuations (\(\rho_0^{\text{HS}}\) and \(\Delta_0^{\text{HS}}\)) since they do not play an important role when the interaction is attractive.

A. Mean-field theory

The mean-field theory is obtained from a saddle-point approximation where the fields \(\rho_0, \Delta_0, \rho_0^{\text{HS}}, \Delta_0^{\text{HS}}\) are taken space and time independent. The saddle-point equations read

\[
\rho_0 = \langle c_r^\dagger c_r \rangle, \\
\rho_0^{\text{HS}} = \frac{g}{2} \rho_0, \\
\Delta_0 = \langle c_r c_{r+} \rangle, \\
\Delta_0^{\text{HS}} = g \Delta_0,
\]
\[
\Delta_0^* = (c_r^* c_r^*), \quad i\Delta_0^{HS} = g\Delta_0^*.
\]

With no loss of generality, we can take \(\Delta_0 = \Delta_0^*\) real. \(i\Delta_0^{HS} = i\Delta_0^{HS*}\) is then real at the saddle point. It is convenient to redefine \(i\Delta_0^{HS} \to \Delta_0^{HS}\) and \(\Delta_0 \to \Delta_0^{HS}\) (so that \(\Delta_0^{HS} = \Delta_0^{HS*}\) is real) and absorb \(\rho_0^{HS}\) in the definition of the chemical potential. The mean-field action is then (up to an additive constant)
\[
S_{MF} = \int_0^\beta d\tau \int d^3r \left[ c_r^\dagger \left( \partial_\tau - \mu - \frac{\nabla^2}{2m} \right) c_r - \Delta_0^{HS} (c_r^* c_r^* + c_r c_r^\dagger) \right] + c.c.
\]

From Eq. (6), we readily obtain the normal and anomalous Green functions
\[
G(k, i\omega) = -(c_r^\dagger(k, i\omega)c_r(k, i\omega)) = -\frac{i\omega - \xi_k}{\omega^2 + E_k^2},
\]
\[
F(k, i\omega) = -(c_r^\dagger(k, i\omega)c_r(-k, -i\omega)) = \frac{\Delta_0^{HS}}{\omega^2 + E_k^2},
\]
where \(E_k = (\xi_k^2 + \Delta_0^{HS^2})^{1/2}\), \(\xi_k = \epsilon_k - \mu\), and \(\sigma = -\sigma\). \(c_r^\dagger(k, i\omega)\) is the Fourier transformed field of \(c_r\) and \(\omega\) a fermionic Matsubara frequency. Using Eqs. (2) and (7), we can rewrite the saddle-point Eqs. (5) as
\[
\frac{m}{4\pi a} = \int_k \frac{1 - \xi_k}{1 - 2E_k},
\]
\[
\rho_0 = \int_k \frac{1 - \xi_k}{E_k},
\]
where \(\Delta_0 = \int d^3k/(2\pi)^3\). Equations (8) determine the chemical potential \(\mu\) and the order parameter \(\Delta_0^{HS} = g\Delta_0\). In the strong-coupling limit \(\rho_0 a^3 \ll 1\), one obtains (see Appendix B)
\[
\mu = -\frac{1}{2ma^2}(1 - 2\rho_0 a^3),
\]
\[
\Delta_0^{HS} = \left( \frac{4\pi \rho_0}{m^2} \right)^{1/2} \left( 1 + \frac{\pi}{4\rho_0 a^3} \right).
\]

### B. Low-energy effective action

In this section, we derive the low-energy effective action for the physical fields \(\rho_0\) and \(\Delta_r\). Since our derivation partially follows Ref. [19], we describe only the main steps (technical details are given in Appendix A). The main assumption is that the collective bosonic fields \(\rho_0, \rho_0^{HS}, \Delta_r\), and \(\Delta_r^{HS}\) weakly fluctuate about their mean-field values.

Starting from the action (4) (with \(\alpha = \gamma = 1\)), where
\[
\Delta_r = |\Delta_r| e^{i\theta_r},
\]
we perform the change of variables
\[
c_r \rightarrow c_r e^{i\theta_r}, \quad \Delta_r \rightarrow e^{i\theta_r}. 
\]

We then consider the shift \(\rho_0^{HS} \to \rho_0^{HS} + i\Delta_r^{HS}\), \(\Delta_0^{HS} \to \Delta_0^{HS} + i\Delta_r^{HS}\) (recall that a factor \(i\) has been included in \(\Delta_0^{HS}\) and \(\Delta_0\)), so that the Hubbard-Stratonovich fields \(\rho_0^{HS}\) and \(\Delta_r^{HS}\) now describe (small) fluctuations about the mean-field values. This leads to the action
\[
S = S_{MF} + \int_0^\beta d\tau \int d^3r \left[ -\frac{\mu_B}{2} - i\rho_0^{HS} \right] - i(\Delta_r^{HS} c_r c_r^* + c.c.) + (i\Delta_r^{HS} + i\Delta_r^{HS*})
\]
\[
+ 2\Delta_0^{HS}|\Delta_r|^2 - \frac{g}{4}\rho_0^2 + (i\rho_0^{HS} + i\rho_0^{HS*})\right],
\]
where \(\tilde{\nabla} = \nabla - \nabla\). Here we write the chemical potential as \(\mu = \mu_{MF} + \mu_B/2\) where \(\mu_{MF}\) is the chemical potential in the mean-field approximation. The next step is to shift \(\rho_0^{HS}, \rho_0^{HS*} \to \rho_0^{HS} + i\theta_r/2 + (\nabla \theta_r)^2/8m - \mu_B/2\), and to introduce Nambu spinors \(\phi_r = (c_r^*, c_r^\dagger)\). This gives
\[
S = S_{MF} + S' + \int_0^\beta d\tau \int d^3r \left[ -g|\Delta_r|^2 - \frac{g^2}{4}\rho_0^2 + (\Delta_0^{HS} + i\Delta_r^{HS})
\]
\[
+ i\Delta_r^{HS*})|\Delta_r|^2 + \rho_0 \left[i\rho_0^{HS} + i\rho_0^{HS*} + i\nabla \theta_r^2 + \frac{\mu_B}{2} \right].
\]

where
\[
S' = \int_0^\beta d\tau \int d^3r \left[ -i\rho_0^{HS} \tilde{\nabla} \tilde{\theta}_r + \frac{1}{2} \tilde{\nabla} \tilde{\theta}_r \tilde{\nabla} \tilde{\theta}_r + \frac{\mu_B}{2} \right].
\]

(\(\tau_\nu, \tau_\tau, \tau_\sigma\) are Pauli matrices acting in Nambu space. The effective action \(S[\rho, \rho_0^{HS}, \Delta, \Delta^{HS}]\) is obtained by integrating out the fermions. To quadratic order in the bosonic fields and their gradient (\(\partial_\tau\) or \(\nabla\)), it is sufficient to retain the first and second-order cumulants of \(S'\) with respect to the mean-field action
where the averages $\langle \cdots \rangle_c$ are calculated with respect to the mean-field action $S_{MF}$. Calculating the first- and second-order cumulants and integrating out the Hubbard-Stratonovich fields $\rho HS^*$ and $\Delta HS^*$ (Appendix A), we obtain

$$S[\rho, \Delta] = \int_0^\beta d\tau \int d^3r \left[ \rho_0 \left( i \frac{\theta_\tau + \nabla \theta_\tau^2}{16} - \frac{\mu_B}{2} \right) \right]$$

$$+ \sum_q \left( \delta \rho_{-q} \delta |\Delta_q| \right) \left( \beta_q - g \alpha_q \right) \left( \delta \rho_{-q} \right),$$

where $\delta \rho_q$ and $\delta |\Delta_q|$ are the Fourier transforms of $\delta \rho_\tau = \rho_\tau - \rho_0$ and $\delta |\Delta_\tau| = |\Delta_\tau| - \Delta_0$, and

$$\beta_q = \frac{1}{2} \Pi_{\rho 0}^\rho(q)^{-1} - \frac{g}{4} - \frac{1}{C_q} \Pi_{\rho 0}^\rho(q)^{-1} \Pi_{\rho 0}^\rho(q)^2.$$

$$- g \alpha_q = \frac{1}{C_q} \Pi_{\rho 0}^\rho(q)^{-1} \Pi_{\rho 0}^\rho(q).$$

$$g^2 \gamma_q = \frac{1}{C_q} - g.$$

$$C_q = - \Pi_{\rho 0}^\rho(q) - \Pi_{\rho 0}^\rho(q) + 2 \Pi_{\rho 0}^\rho(q)^{-1} \Pi_{\rho 0}^\rho(q)^2. \tag{18}$$

We use the notation $q = (q, \omega_q)$ and $\Sigma_q = \Sigma_{q, \omega_q}$ where $\omega_q$ is a bosonic Matsubara frequency. The mean-field correlation function $\Pi^\rho(q) = (j^\rho_\tau(q)j^\rho_\tau(-q))_c$ is calculated in Appendix B and $\Pi^\rho(q) = \Pi^{\rho*}(q, \omega_q = 0)$. $j^\rho_\tau(q)$ is the Fourier transformed field of $j^\rho_\tau$. Equations (15) and (17) shows that half the fermion density is the conjugate variable of the phase $\theta_\tau$ of the pairing field. Equations (17) and (18) agree with Eq. (2.3) of Ref. [19] except for the coefficient of $\delta \rho_{-q} \delta |\Delta_q|$ which is found to have opposite sign [26].

We now discuss the strong-coupling limit (not considered in Ref. [19]). To leading order in $\rho_0 a^3$ and $|\mathbf{q}| a$, we have (Appendix B) [27]

$$\alpha_q = \left( \frac{1}{4 \pi a} \right)^{1/2} \left( 1 + \frac{9}{4} \pi a^3 + \frac{1}{6} |\mathbf{q}|^2 a^2 \right).$$

$$\beta_q = \left( \frac{1}{2 \rho_0 a^2} \right) \left( 1 + 4 \pi a^3 + \frac{1}{4} |\mathbf{q}|^2 a^2 \right).$$

$$\gamma_q = \frac{m}{2 \pi a} \left( 1 + \frac{3}{2} \pi a^3 + \frac{7}{48} |\mathbf{q}|^2 a^2 \right). \tag{19}$$

Denoting by $\lambda^+ q$ and $\lambda^- q$ the two eigenvalues of the fluctuation matrix appearing in Eq. (17), we have

$$\lambda^+ = \beta_q + g^2 \alpha_q^2 \beta_q,$$

$$\lambda^- = g^2 \left( \gamma_q - \frac{\alpha_q}{\beta_q} \right). \tag{20}$$

To order $O(g^2)$. For $g \rightarrow 0$ (at fixed $a$), the mode corresponding to the eigenvalue $\lambda^+ q$ is frozen, which leads to

$$\frac{\delta \lambda^+}{\delta \rho_0} = \frac{1}{g} \left( \frac{\pi}{\rho_0 a^2} \right)^{1/2}. \tag{21}$$

Density ($\delta \rho_0$) and modulus ($\delta \Delta_\tau$) fluctuations do not fluctuate independently in the low-energy limit but are tied by the relation (21). From Eqs. (17), (19), and (21), we deduce that the dynamics of the Fermi superfluid is determined by the effective action

$$S[\rho, \theta] = \int_0^\beta d\tau \int d^3r \left[ \rho_0 \left( i \frac{\theta_\tau + \nabla \theta_\tau^2}{16} - \frac{\mu_B}{2} \right) + \frac{\pi a}{2 \rho_0 m^2} \left( \nabla \rho_0 \right)^2 \right.$$

$$+ \frac{\nabla \rho_0}{2 \rho_0 m^2} \left( \frac{\rho_0}{\rho_0 m^2} \right) \left. \right]. \tag{22}$$

Introducing the bosonic field

$$\psi_\tau = \sqrt{\frac{\rho_0}{2}} e^{i \theta_\tau}, \tag{23}$$

we recover the standard action of a Bose superfluid,

$$S[\psi^*, \psi] = \int_0^\beta d\tau \int d^3r \left[ \psi^* \left( \partial_\tau - \mu_B - \frac{\nabla^2}{2 \rho_0 m} \right) \psi + \frac{2 \pi a}{m_B} \left( \psi^* \psi \right) \right.$$

$$\left. - \rho_0 / 2 \right] \right]. \tag{24}$$

where $m_B = 2 m$ and $a_B = 2 a$ are the mass and the scattering length of the bosons. The result $a_B = 2 a$ corresponds to the Born approximation for the boson-boson scattering, while the exact result is $a_B = 0.6 a$ [28]. Equations (24) and (22) are equivalent in the hydrodynamic regime where $\left( \nabla \rho_0 \right)^2 / \rho_0 = (\nabla \rho_0)^2 / \rho_0$ [29].

Thus, we have shown how, by introducing the physical fields $\rho_\tau$ and $\Delta_\tau$ from the outset and expanding about the mean-field state in the strong-coupling limit, one obtains the standard action of a Bose superfluid. Our approach should be contrasted with a number of previous works [6,13–18] where only the pairing Hubard-Stratonovich field $\Delta_\tau$ is considered and the expansion is carried out about the non-interacting state, which gives the action (24) but for the field $\sqrt{\rho_\tau / 2 \sin \theta_\tau}$ instead of the $\psi$ field defined in (23) [30].
III. LATTICE MODEL

In this section, we consider the attractive Hubbard model on a bipartite lattice, with Hamiltonian

\[ H = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} (c_{\mathbf{r}}^\dagger c_{\mathbf{r}'} + h.c.) - \mu \sum_{\mathbf{r}} c_{\mathbf{r}}^\dagger c_{\mathbf{r}} - U \sum_{\mathbf{r}} n_{\mathbf{r}c} n_{\mathbf{r}c} . \]

(25)

The operator \( c_{\mathbf{r}c}^\dagger c_{\mathbf{r}c} \) creates (annihilates) a fermion with spin \( \sigma \) at the lattice site \( \mathbf{r} \), \( c_{\mathbf{r}} = (c_{\mathbf{r}c}, c_{\mathbf{r}c}^\dagger)^T \), and \( n_{\mathbf{r}c} = c_{\mathbf{r}c}^\dagger c_{\mathbf{r}c} \). \( \langle \mathbf{r}, \mathbf{r}' \rangle \) denotes nearest-neighbor sites. The chemical potential \( \mu \) fixes the average density \( \rho_0 \) (i.e., the average number of fermions per site) and \(-U(U \geq 0)\) is the on-site attractive interaction.

We are interested in the strong-coupling limit \( U \gg t \) where fermions form tightly bound composite bosons which behave as local pairs. The latter Bose condense at low temperature giving rise to superfluidity. In order to derive the low-energy effective action, we could follow the procedure used in Sec. II. Here, we shall use a different method, based on the mapping of the attractive Hubbard model in the strong-coupling limit onto the Heisenberg model in a uniform magnetic field [20]. Thus this approach is based on a t/U expansion about the \( t=0 \) limit rather than on an expansion about the mean-field state [31].

Under the canonical particle-hole transformation [32]

\[ c_{\mathbf{r}c} \rightarrow (-1)^\mathbf{r} c_{\mathbf{r}c}^\dagger, \quad c_{\mathbf{r}c}^\dagger \rightarrow (-1)^\mathbf{r} c_{\mathbf{r}c}, \]

(26)

the Hamiltonian becomes (omitting a constant term)

\[
H = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} (c_{\mathbf{r}}^\dagger c_{\mathbf{r}'} + h.c.) - \sum_{\mathbf{r}} c_{\mathbf{r}}^\dagger \left( \frac{U}{2} + h_0 \sigma \right) c_{\mathbf{r}} \\
+ U \sum_{\mathbf{r}} n_{\mathbf{r}c} n_{\mathbf{r}c},
\]

(27)

and corresponds now to the repulsive Hubbard model in a magnetic field \( h_0 = h_0 \mathbf{\hat{z}} \) along the \( z \) axis,

\[ h_0 = \mu + \frac{U}{2}, \]

(28)

coupled to the fermion spins. The chemical potential \( U/2 \) in Eq. (27), together with particle-hole symmetry, implies that the system is half-filled. The density and pairing operators transform into the three components of the spin density operator

\[ \rho_\tau = c_{\mathbf{r}c} c_{\mathbf{r}c}^\dagger \rightarrow c_{\mathbf{r}c}^\dagger \sigma^\tau c_{\mathbf{r}c} + 1, \]

\[ \Delta_\tau = c_{\mathbf{r}c} c_{\mathbf{r}c}^\dagger \rightarrow (-1)^\mathbf{r} c_{\mathbf{r}c}^\dagger \sigma^\tau c_{\mathbf{r}c}, \]

\[ \Delta^\tau = c_{\mathbf{r}c}^\dagger c_{\mathbf{r}c} \rightarrow (-1)^\mathbf{r} c_{\mathbf{r}c} c_{\mathbf{r}c}^\dagger. \]

(29)

The equation fixing \( \rho, \langle c_{\mathbf{r}c}^\dagger \sigma^\tau c_{\mathbf{r}c} \rangle = \rho_0 \), becomes an equation fixing the magnetic field: \( \langle c_{\mathbf{r}c}^\dagger \sigma^\tau c_{\mathbf{r}c} \rangle = \rho_0 - 1 \).

In the strong-coupling limit \( U \gg t \), the Hamiltonian (27) simplifies into [20]

\[ H = J \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \mathbf{S}_\mathbf{r} \cdot \mathbf{S}_{\mathbf{r}'}, \]

(30)

where \( J = 4t^2/U \) and \( \mathbf{S}_\mathbf{r} \) is a spin-1/2 operator. Using spin-1/2 coherent states \( \langle \Omega_\tau = 0 \rangle \) [33], the action of the Heisenberg model (30) can be written as

\[ S[\Omega] = \int_0^\beta d\tau \sum_{\mathbf{r}} \left( \langle \Omega_{\mathbf{r}} \rangle - h_0 \cdot \mathbf{\Omega}_{\mathbf{r}} \right) + J \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \frac{\mathbf{\Omega}_{\mathbf{r}} \cdot \mathbf{\Omega}_{\mathbf{r}'}}{4}, \]

(31)

where \( \mathbf{\Omega}_{\mathbf{r}} = \partial_\tau \mathbf{\Omega}_{\mathbf{r}} \).

The effective action \( S[\rho, \Delta] \) of the superfluid system is obtained by rewriting the action (31) in terms of the density and pairing fields of the attractive model. In the strong-coupling limit, Eqs. (29) (written now for fields rather than operators) become [20]

\[ \rho_\tau = \rho^\tau + 1, \]

\[ \Delta_\tau = \frac{(1)^\mathbf{r}}{2} \mathbf{\Omega}_-^\tau, \]

\[ \Delta^\tau = \frac{(1)^\mathbf{r}}{2} \mathbf{\Omega}_+^\tau, \]

where \( \mathbf{\Omega}_+^\tau = \mathbf{\Omega}_+^\tau \mp i \mathbf{\Omega}_-^\tau \). The condition \( \mathbf{\Omega}_+^2 = 1 \) implies that \( \rho_\tau \) and \( \Delta_\tau \) do not fluctuate independently but are tied by the relation

\[ |\Delta_\tau| = \frac{1}{2} \left( \rho_\tau (2 - \rho_\tau) \right)^{1/2}. \]

(33)

In the low-density limit \( (\rho_\tau \ll 1) \), where the Pauli principle (which prevents two composite bosons to occupy the same site) should not matter, we expect to recover the standard action of a Bose superfluid. In that limit, \( |\Delta_\tau| = \sqrt{\rho_\tau / 2} \); the pair density \( |\Delta_\tau|^2 \) equals half the fermion density \( \rho_0 \), and \( \Delta_\tau = |\Delta_\tau| e^{i \theta_\tau} \) coincides with the bosonic field \( \psi_\tau = \sqrt{\rho_\tau / 2} e^{i \theta_\tau} \). To order \( \mathcal{O}(\rho_\tau^2) \), we deduce from Eqs. (31)–(33)

\[ S[\rho, \theta] = \int_0^\beta d\tau \left\{ \sum_{\mathbf{r}} \left[ \frac{i}{2} \rho_\tau \theta_\tau - \left( h_0 + \frac{J_z}{4} \right) \rho_\tau \right] + \frac{J}{4} \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \right. \left[ \rho_\tau \rho_{\mathbf{r}'}, \right. \left. \right. \left. \left. - (\rho_\tau \rho_{\mathbf{r}'})^{1/2} (2 - \rho_\tau) \cos(\theta_\tau - \theta_{\mathbf{r}'}) \right] \right\}. \]

(34)

The term \( (i/2) \rho_\tau \theta_\tau \) comes from the Berry phase term \( \langle \Omega_{\mathbf{r}c} \rangle \) of the action \( S[\Omega] \) [Eq. (31)] with a proper gauge choice [20]. If we further assume that \( \rho_\tau \) and \( \theta_\tau \) are slowly varying in space, we obtain

\[ S[\rho, \theta] = \int_0^\beta d\tau \sum_{\mathbf{r}} \left\{ \frac{i}{2} \rho_\tau \theta_\tau - \left( h_0 + \frac{J_z}{4} \right) \rho_\tau + \frac{J_z}{4} \rho_\tau^2 \right\} - \frac{J}{2} \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \left( \rho_\tau \rho_{\mathbf{r}'}, \right)^{1/2} \cos(\theta_\tau - \theta_{\mathbf{r}'}) \right\}. \]

(35)

or, equivalently
\[ S[\psi', \bar{\psi}] = \int_0^\beta d\tau \left\{ \sum_r \left[ \psi_r^*(\partial_\tau - \mu)\psi_r + \frac{U_B}{2}|\psi_r|^4 \right] \right\} - t_B \sum_{(r,r')} (\psi_r \psi_{r'} + c.c.), \tag{36} \]

where \( t_B = J/2, U_B = 2J_z, \mu_B = 2h_0 + J_z/2, \) and \( z \) is the number of nearest-neighbor sites. We therefore obtain the action of the Bose-Hubbard model with on-site repulsive interaction \( U_B \) and nearest-neighbor hopping amplitude \( t_B \). In the continuum limit and for a cubic lattice, the latter gives a boson mass \( m_B = 1/J \) as obtained in Ref. [20].

IV. CONCLUDING REMARKS

In this paper, we have shown that a Fermi superfluid in the strong-coupling limit, where superfluidity originates from BEC of composite bosons, can be described by the complex field \( \psi_r = \sqrt{\rho_r} e^{i \theta_r} \), where \( \rho_r \) is the fermion density and \( \theta_r \) the phase of the pairing field \( \Delta_r \). Such description is made possible by the fact that \( \rho_r \) and amplitude \( \langle |\Delta_r| \rangle \) fluctuations are not independent in the strong-coupling limit. The effective action \( S[\rho, \theta] \) is derived by introducing the physical fields \( \rho_r \) and \( \Delta_r \) from the outset by means of Lagrange multiplier fields \( \rho_r^{HS} \) and \( \Delta_r^{HS} \). The latter play the role of the Hubbard-Stratonovich fields usually introduced via a Hubbard-Stratonovich transformation of the fermion-fermion interaction.

For continuum models, the effective action is derived from an expansion about the mean-field state. It corresponds to the usual action of a Bose superfluid of density \( \rho_r/2 \) where the bosons have a mass \( m_B = 2m \) and interact via a contact potential with amplitude \( g = 4 \pi a_B/m_B, a_B = 2a \).

For lattice (Hubbard) models, the effective action is obtained from an expansion about the \( r = 0 \) limit, using the mapping of the attractive Hubbard model in the strong-coupling limit onto the Heisenberg model in a uniform magnetic field. The effective model is a Bose-Hubbard model with an on-site repulsion \( U_B = 2J_z \) (with \( z \) the number of nearest-neighbor sites) and a nearest-neighbor intersite hopping amplitude \( t_B = J/2 \), where \( J = 4t^2/U \).

APPENDIX A: LOW-ENERGY EFFECTIVE ACTION \( S[\rho, \Delta] \)

In this Appendix, we derive the effective action (17) starting from Eq. (16). The first- and second-order cumulants are given by

\[ \langle S' \rangle = \int_0^\beta d\tau \int d^3r \left[ -i \rho_r^{HS} - i \Delta_r (\Delta_r^{HS} + \Delta_r^{HS}) \right], \]

where averages \( \langle \cdots \rangle_0 \) are taken with the Gaussian action

\[ S_0[\Delta^{HS}] = -\frac{1}{2} \sum_q (\Delta_r^{HS} + \Delta_q^{HS}) M^{-1}(q) \left( \frac{\Delta_q^{HS}}{\Delta_r^{HS}} \right), \]

\[ M^{-1}(q) = \left( \begin{array}{cc} -\Pi^{zz}_0(q) + \Pi^{zz}_0(q)^{-1} \Pi^{zz}_0(q)^2 & -\Pi^{zz}_0(q) + \Pi^{zz}_0(q)^{-1} \Pi^{zz}_0(q)^2 \\ -\Pi^{zz}_0(q) + \Pi^{zz}_0(q)^{-1} \Pi^{zz}_0(q)^2 & -\Pi^{zz}_0(q) + \Pi^{zz}_0(q)^{-1} \Pi^{zz}_0(q)^2 \end{array} \right). \tag{A4} \]
In the following, we denote by $A_q$ and $B_q$ the diagonal and off-diagonal components of $M^{-1}(q)$, and $C_q = A_q + B_q$. The effective action $S[\rho, \Delta]$ deduced from Eqs. (A2) and (A3) is given by Eq. (17).

APPENDIX B: MEAN-FIELD CORRELATION FUNCTION

In this Appendix, we calculate the mean-field correlation function $\Pi_{\mu\nu}(q) = \langle j_{\mu}^0(q) j_{\nu}^0(-q) \rangle \langle \nu, \nu' = x, y, z; \mu, \mu' = 0, x, y, z \rangle$ in the strong-coupling limit $\rho_0 a^3 \ll 1$ and for $|q|a \ll 1$.

1. General expression

$j_{\mu}^0(q)$ is the Fourier transformed field of $j_{\mu\nu}$ [Eq. (15)]:

$$j_{\mu}^0(q) = \frac{1}{\sqrt{V}} \sum_{k} \left( k_{\mu} + \frac{q_{\mu}}{2} \right) \phi_{k} \phi_{k+q} (\mu \neq 0),$$  \hspace{1cm} (B1)

where $V$ is the volume of the system and $V^{-1} \Sigma_k = S_k$ for $V \rightarrow \infty$. $k = (k, \mathbf{q}, \omega)$ and $\Sigma_k = \Sigma_k(\omega)$ where $\omega$ is a fermionic Matsubara frequency. We have

$$\Pi_{00}^0(q) = -\frac{2}{V} \sum_{k} \left[ G(k) G(k+q) - F(k) F(k+q) \right],$$

$$\Pi_{00}^0(q) = \Pi_{00}^0(-q) = -\frac{2}{V} \sum_{k} G(k+q) F(k),$$

$$\Pi_{00}^0(q) = \frac{1}{V} \sum_{k} F(k) F(k+q),$$

$$\Pi_{00}^0(q) = \frac{1}{V} \sum_{k} G(k) G(-k-q),$$

$$\Pi_{00}^0(q) = -\frac{2}{V} \sum_{k} \left( k_{\mu} + \frac{q_{\mu}}{2} \right) \left( k_{\mu'} + \frac{q_{\mu'}}{2} \right)$$

$$\times \left[ G(k+q) G(k+q) + F(k+q) F(k) \right],$$  \hspace{1cm} (B2)

where $G$ and $F$ are the mean-field propagators [Eq. (7)]. The correlation function $\langle j_{\mu}^0(q) j_{\nu}^0(-q) \rangle (\mu \neq 0)$ vanishes.

In the following, we consider the static limit $\Pi_{00}^{\mu\nu}(q) = \Pi_{00}^{\nu\mu}(q, \omega = 0)$. Performing the sum over Matsubara frequency in Eq. (B2) in the $T = 0$ limit, we obtain

$$\Pi_{00}^{0+}(q) = \frac{1}{V} \sum_{k} \left( 1 - \frac{E_k}{E_k + E_{k+q}} \right),$$

$$\Pi_{00}^{0-}(q) = \frac{1}{V} \sum_{k} \frac{E_k}{E_k + E_{k+q}}.$$

The correlation function $\Pi_{00}^{\mu\nu}(q)$ vanishes for $q = 0$. Since $j_{\mu}^0$ multiplies $\phi_{k} \phi_{k+q}$ in the action $\mathcal{S}$, it is sufficient to consider $\Pi_{00}^{0\mu\nu}(q) = 0$ to obtain the effective action $S[\rho, \Delta]$ to order $(\phi_{k} \phi_{k+q})^2$.

We next expand the correlations to order $|q|^2$. Writing $\xi_k = \xi_k + X_{k,q}$ with $X_{k,q} = k \cdot q/m + |q|^2/2m$, we obtain

$$\Pi_{00}^{0+}(q) = \int \frac{d^3k}{(2\pi)^3} \frac{E_k}{E_k + E_{k+q}} \left( 1 + \frac{3\xi_k^2}{2E_k} \right) X_{k,q},$$

$$\Pi_{00}^{0-}(q) = \int \frac{d^3k}{(2\pi)^3} \frac{E_k X_{k,q}}{E_k + E_{k+q}} \left( 1 - \frac{3\xi_k^2}{2E_k} \right),$$

$$\Pi_{00}^{0}(q) = \int \frac{d^3k}{(2\pi)^3} \frac{E_k X_{k,q}}{E_k + E_{k+q}} \left( 1 + \frac{3\xi_k^2}{2E_k} \right).$$  \hspace{1cm} (B3)

2. Strong-coupling limit $\rho_0 a^3 \ll 1$

In the strong-coupling limit, the chemical potential $\mu$ is negative. We then have

$$\int \frac{1}{E_k} = \frac{m \Lambda}{\pi} = \frac{m^{3/2} |\mu|^{1/2}}{\sqrt{2} \pi},$$

$$\int_0^\infty \frac{k^4}{\xi_k^2} = \frac{3\pi m^{5/2}}{2 \sqrt{2} |\mu|^{1/2}}.$$

Other useful relations are obtained by differentiating Eqs. (B5) with respect to $\mu$. Note that $\Lambda$ is sent to infinity whenever the integral over $k$ converges. In the strong-coupling limit, the small parameter expansion is $\Delta^{HS}_0/|\mu|^2 \sim \rho_0 a^3$. Approximate expressions of the mean-field correlation functions can be obtained by expanding Eqs. (8) and (B4) in power of $\Delta^{HS}_0$ and using Eqs. (B5) [as well as those obtained from (B5) by differentiating with respect to $\mu$]. A straightforward (but somewhat lengthy) calculation then gives Eq. (9) and
\[ \Pi^{\text{sc}}_{00}(\mathbf{q}) = \frac{\rho_0ma^2}{4} \left( 1 - 4\pi\rho_0a^3 - \frac{|\mathbf{q}|^2a^2}{4} \right), \]
\[ \Pi^{\text{ch}}_{00}(\mathbf{q}) = \frac{(\rho_0m^2a^4)}{4\pi} \left( 1 - \frac{7}{4} \pi\rho_0a^3 - \frac{|\mathbf{q}|^2a^2}{12} \right), \]
\[ \Pi^{\text{ch}}_{00}(\mathbf{q}) = -\frac{\rho_0ma^2}{4} \left( 1 - 4\pi\rho_0a^3 - \frac{5}{16} |\mathbf{q}|^2a^2 \right), \]
\[ \Pi^{\text{ch}}_{00}(\mathbf{q}) = -\frac{1}{g} \frac{\rho_0ma^2}{4} \left( 1 - 4\pi\rho_0a^3 \right) - \frac{ma|\mathbf{q}|^2}{32\pi}. \]