Collective modes in a system with two spin-density waves: The Ribault phase of quasi-one-dimensional organic conductors

N. Dupuis*
Laboratoire de Physique des Solides, Université Paris-Sud, 91405 Orsay, France and Department of Physics, University of Maryland, College Park, Maryland 20742-4111

Victor M. Yakovenko
Department of Physics, University of Maryland, College Park, Maryland 20742-4111

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We study the long-wavelength collective modes in the magnetic-field-induced spin-density-wave (FISDW) phases experimentally observed in organic conductors of the Bechgaard salts family, focusing on phases that exhibit a sign reversal of the quantum Hall effect (Ribault anomaly). We have recently proposed that two SDW's coexist in the Ribault phase, as a result of umklapp processes. When the latter are strong enough, the two SDW's become circularly polarized (helicoidal SDW's). In this paper, we study the collective modes that result from the presence of two SDW's. We find two Goldstone modes, an out-of-phase sliding mode and an in-phase spin-wave mode, and two gapped modes. The sliding Goldstone mode carries only a fraction of the total optical spectral weight, which is determined by the ratio of the amplitude of the two SDW's. In the helicoidal phase, all the spectral weight is pushed up above the SDW gap. We also point out similarities with phase modes in two-band, bilayer, or $d+id'$ superconductors. We expect our conclusions to hold for generic two-SDW systems.

I. INTRODUCTION

In electron systems with broken symmetries, such as superconductors or density-wave (DW) systems, quasiparticle excitations are often gapped, and the only low-lying excitations are collective modes. The latter thus play a crucial role in various low-energy properties.

In an incommensurate spin-density-wave (SDW) system, there are two (gapless) Goldstone modes: a sliding mode and a spin-wave mode, which result from the spontaneous breaking of translation symmetry in real space and rotation symmetry in spin space, respectively.[1,2] Contrary to the case of superconductors, collective modes in DW systems directly couple to external probes and therefore show up in various experiments. For instance, the sliding mode, which is pinned by impurities in real systems, can be depinned by a strong electric field. This leads to nonlinear conduction, observed in DW systems.[1]

The aim of this article is to study long-wavelength collective modes in a quasi-one-dimensional (quasi-1D) system where the low-temperature phase exhibits two SDW's. The presence of two SDW's gives rise to a rich structure of collective modes, which in principle can be observed in experiments.[3]

Our results are based on a particular case: the magnetic-field-induced spin-density-wave (FISDW) phases of the organic conductors of the Bechgaard salt family.[4–7] These FISDW phases share common features with standard SDW phases, but also exhibit remarkable properties like the quantization of the Hall effect. We have shown that umklapp processes may lead in these systems to the coexistence of two SDW's with comparable amplitudes, which provides a possible explanation of the sign reversal of the quantum Hall effect (QHE) (the so-called Ribault anomaly[8,9]) observed in these conductors. There are two motivations for studying this particular case. (i) The Bechgaard salts, as possible candidates for a two-SDW system, present their own interest. (ii) A conductor with two SDW's is in general not easy to analyze, even at the mean-field level.[10] The analysis simplifies when a strong magnetic field quantizes the electron motion.

Nevertheless, we expect our conclusions to be quite general and to apply (at least qualitatively) to other systems with two SDW's. This hope is strongly supported by the similarities[3] that exist between collective modes in two-SDW conductors and phase modes in two-band,[11] bilayer,[12] or $d+id'$ (Ref. 13) superconductors, and, to a lesser extent, plasmon modes in semiconductor double-well structures.[14] These similarities suggest that collective modes in two-component systems present generic features that do not depend on the particular case considered.

A. Umklapp processes in quasi-1D SDW systems

Consider a quasi-1D conductor with a SDW ground state. In the presence of umklapp processes transferring momentum $\mathbf{K}$ ($\mathbf{K}$ being a vector of the reciprocal space), spin fluctuations at wave vectors $\mathbf{Q}$ and $\mathbf{K}-\mathbf{Q}$ are coupled. Thus, the formation of a SDW at wave vector $\mathbf{Q}_1$ will automatically be accompanied by the formation of a second SDW at wave vector $\mathbf{Q}_2=\mathbf{K}-\mathbf{Q}_1$, provided that $\mathbf{Q}_1\neq\mathbf{Q}_2$. The case $\mathbf{Q}_1=\mathbf{Q}_2=\mathbf{K}/2$ corresponds to a (single) commensurate SDW. Umklapp processes pin the SDW whose position with respect to the underlying crystal lattice becomes fixed: the sliding mode is gapped. For two incommensurate SDW's ($\mathbf{Q}_1\neq\mathbf{Q}_2$),[15] the total spin-density modulation can then be written as

$$\langle \mathbf{S}(\mathbf{r})\rangle=\sum_{i=1,2}\mathbf{S}_i\cos(\mathbf{Q}_i\cdot\mathbf{r}+\theta_i), \tag{1.1}$$

where $\mathbf{r}=(na,mb)$ (with n,m integers) denotes the position in real space (a and b being the lattice spacings along and across the conducting chains).

Even for $\mathbf{Q}_1 \neq \mathbf{Q}_2$, the distinction between the two SDW's may appear somewhat unjustified since one cannot distinguish between $\cos(\mathbf{Q}_1 \cdot \mathbf{r})$ and $\cos(\mathbf{Q}_2 \cdot \mathbf{r})$ when $\mathbf{r}$ is taken as a discrete variable. However, umklapp processes do lead to the presence of two nonvanishing order parameters, $\langle c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}+\mathbf{Q}_1\downarrow}\rangle$ and $\langle c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}+\mathbf{Q}_2\downarrow}\rangle$, in the SDW phase.[16] This doubles the number of degrees of freedom of the SDW condensate, which yields, for instance, twice as many collective modes (as compared to the case with a single SDW). Thus, it is natural to speak of two SDW's in the ground state of the system. Furthermore, we note that $\cos(\mathbf{Q}_1 \cdot \mathbf{r}+\theta_1)$ and $\cos(\mathbf{Q}_2 \cdot \mathbf{r}+\theta_2)$ are indistinguishable only if $\theta_1=-\theta_2$ (with, again, $\mathbf{r}$ being a discrete variable). This is precisely the equilibrium condition obtained by minimizing the mean-field condensation energy (see Sec. III). Condensate fluctuations do not in general satisfy the condition $\theta_1=-\theta_2$, and it is then more appropriate to view these fluctuations as originating from two different SDW's.

In a quasi-1D system with a single SDW, the wave vector $\mathbf{Q}$ of the spin-density modulation is determined by the nesting properties of the Fermi surface: $E(\mathbf{k}+\mathbf{Q})\simeq -E(\mathbf{k})$, where $E(\mathbf{k})$ is the energy with respect to the Fermi level. [For a perfectly nested Fermi surface, one would have $E(\mathbf{k}+\mathbf{Q})=-E(\mathbf{k})$.] Umklapp processes are important only if both $\mathbf{Q}_1$ and $\mathbf{Q}_2=\mathbf{K}-\mathbf{Q}_1$ are good nesting vectors. Otherwise, one of the two SDW's has a very small amplitude and can be ignored for any practical purpose.

For a 2D (or 3D) conductor, the geometry of the Fermi surface appears to be crucial. Consider the following dispersion law, which is linearized in the vicinity of the Fermi level:

$$E_\alpha(k_x,k_y)=v_F(\alpha k_x-k_F)-2t_b\cos(k_y b+\alpha\kappa)+\cdots, \tag{1.2}$$

where k_x and k_y are the electron momenta along and across the conducting chains. $\alpha=+\ (-)$ corresponds to right (left) movers with momenta close to αk_F. v_F and k_F are the Fermi velocity and momentum for the motion along the chains, t_b the interchain transfer integral, and b the interchain spacing. The ellipses in Eq. (1.2) represent small corrections that generate deviations from perfect nesting. κ is a parameter which parametrizes the shape of the Fermi surface.

Most calculations on quasi-1D SDW systems assume $\kappa=0$, which corresponds to the Fermi surface shown in Fig. 1(a). There are two ''best'' nesting vectors: $\mathbf{Q}_1\simeq(2k_F,\pi/b-\delta)$ and $\mathbf{Q}_2\simeq(2k_F,-\pi/b+\delta)$, where the small correction δ ($|\delta|\ll\pi/b$) is due to deviations from perfect nesting.[17,18] Since $\mathbf{Q}_2=(4k_F,0)-\mathbf{Q}_1$, these two vectors are coupled by umklapp scattering if the system is half filled ($4k_F=2\pi/a$). Two SDW's with equal amplitudes will form simultaneously at low temperature.

Consider now the case $\kappa=\pi/4$ in a half-filled band, which corresponds to the asymmetric Fermi surface shown in Fig. 1(b). This Fermi surface has been proposed as a good approximation to the actual Fermi surface of the Bechgaard salts.[19,20] The best nesting vector $\mathbf{Q}_1=(2k_F,\pi/2b-\delta)$ is now nondegenerate. By umklapp scattering, $\mathbf{Q}_1$ couples to $\mathbf{Q}_2=(2k_F,-\pi/2b+\delta)$, which is not a good nesting vector. At low temperature, two SDW's will form simultaneously, but the one with wave vector $\mathbf{Q}_2$ will have a vanishingly small amplitude.

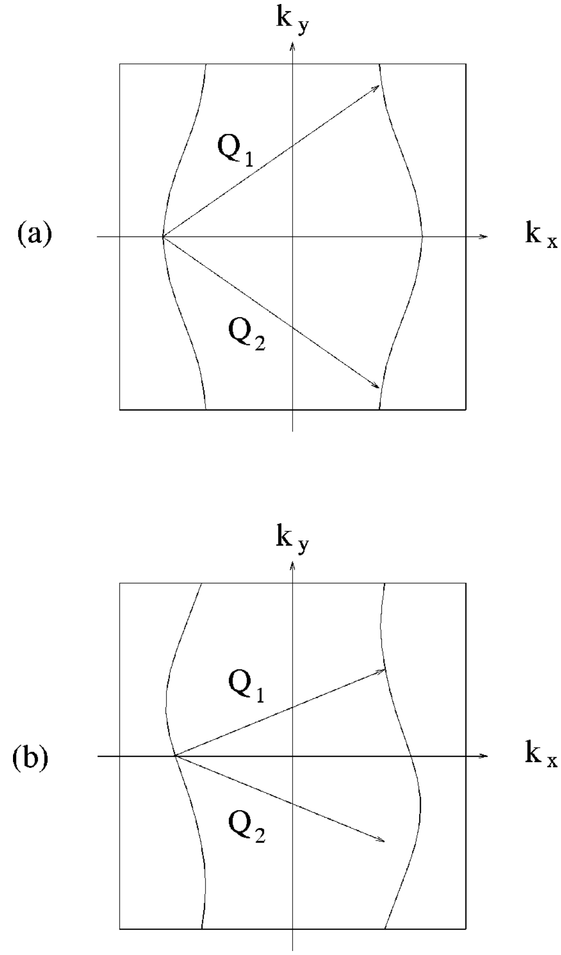

FIG. 1. (a) Fermi surface deduced from the dispersion law (1.2) with $\kappa=0$. At half filling, the two ''best'' nesting vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are coupled by umklapp scattering. (b) Fermi surface for $\kappa=\pi/4$. There is only one ''best'' nesting vector ($\mathbf{Q}_1$). By umklapp scattering, it couples to $\mathbf{Q}_2$, which is not a good nesting vector.

B. The Ribault phase of the Bechgaard salts

The organic conductors of the Bechgaard salts family $(\mathrm{TMTSF})_2X$ (where TMTSF stands for tetramethyltetraselenafulvalene) are well known to have remarkable properties in a magnetic field. In three members of this family ($X=\mathrm{ClO}_4$, PF_6, ReO_4), a moderate magnetic field of a few tesla destroys the metallic phase and induces a series of SDW phases separated by first-order phase transitions.[4,5]

According to the so-called quantized nesting model (QNM),[5] the formation of the magnetic-field-induced spin-density waves results from competition between the nesting properties of the Fermi surface and the quantization of the electron motion in a magnetic field. The formation of a SDW opens a gap, but leaves closed pockets of electrons and/or holes in the vicinity of the Fermi surface. In the absence of a magnetic field, these pockets are too large (due to imperfect nesting) for the SDW phase to be stable. In the presence of a magnetic field H, they become quantized into Landau levels (more precisely, Landau subbands). In each FISDW phase, the SDW wave vector is quantized, $\mathbf{Q}_N=(2k_F+NG,Q_y)$ with N integer, so that an integer number of Landau subbands are filled. (Here $G=eHb/\hbar \ll k_F$ and $-e$ is the electron charge.) As a result, the Fermi level lies in a gap between two Landau subbands, the SDW phase is stable, and the Hall conductivity is quantized: $\sigma_{xy}=-2Ne^2/h$ per one layer of the TMTSF molecules.[6,7] As the magnetic field increases, the value of the integer N changes, which leads to a cascade of FISDW transitions. The QNM predicts the integer N to have always the same sign. While most of the Hall plateaus are indeed of the same sign, referred to as positive by convention, a negative QHE is also observed at certain pressures (the so-called Ribault anomaly).[8,9] The most commonly observed negative phases correspond to $N=-2$ and $N=-4$.

In the Bechgaard salts, a weak dimerization along the chains leads to a half-filled band. Umklapp processes transferring $4k_F=2\pi/a$ are allowed. Thus the formation of a SDW at wave vector $\mathbf{Q}_N=(2k_F+NG,Q_y)$ will be accompanied by a second SDW at wave vector $\mathbf{Q}_{\bar{N}}=(4k_F,0)-\mathbf{Q}_N=(2k_F-NG,-Q_y)$, i.e., there is coexistence of phases N and $-N$.[21] Note that, in our notation, $\mathbf{Q}_{\bar{N}}$ has the signs of both N and Q_y reversed compared to $\mathbf{Q}_N$. As discussed in the preceding section, actual coexistence may occur only if $\mathbf{Q}_{\bar{N}}$ (like $\mathbf{Q}_N$) is a good nesting vector. This is the case with the Fermi surface shown in Fig. 1(a) (since $Q_y\sim\pi/b$), but not with the one shown in Fig. 1(b) (since $Q_y\sim\pi/2b$).

In Ref. 21, we have studied the effect of umklapp scattering on the FISDW phases, starting from the Fermi surface shown in Fig. 1(a). We have shown that for weak umklapp scattering $Q_y\neq\pi/b$. In that case, the SDW with negative quantum number $-|N|$ has a vanishing amplitude and can be ignored. However, for N even, there exists a critical value of the umklapp scattering strength above which the system prefers to form two transversely commensurate SDW's ($Q_y=\pi/b$). For a certain dispersion law,[22] the SDW with negative quantum number $-|N|$ has the largest amplitude, which leads to a negative Hall plateau [Fig. 2(a)]. Since the umklapp scattering strength is sensitive to pressure, we have suggested that this provides a natural explanation for the negative QHE (Ribault anomaly) observed in the Bechgaard salts.[23] (In the following, the "negative" phases are referred to as "Ribault" phases.)

It should be noted that this explanation relies on a simple Fermi surface [Fig. 1(a)], which does not necessarily provide a good approximation to the actual Fermi surface of the Bechgaard salts. With the more realistic (according to band calculations) Fermi surface shown in Fig. 1(b), umklapp processes have only a small effect and do not lead to a negative QHE.[24] Therefore, our explanation of the Ribault anomaly should be taken with caution. For it to be correct, the parameter κ should be smaller (typically not larger than $\pi/10$) than the value $\pi/4$ predicted by band calculation. In the following, we consider only the case $\kappa=0$.

We have shown in Ref. 21 that the SDW's in the Ribault phase are likely to become circularly polarized (helicoidal SDW's) when the umklapp scattering strength is further increased [Fig. 2(b)]. The QHE vanishes in the helicoidal phase. We will see that the circular polarization also affects the collective modes.

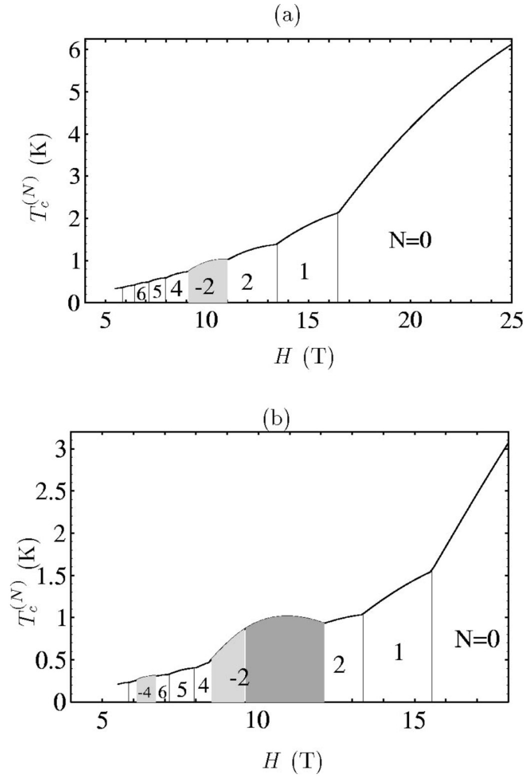

FIG. 2. Transition temperature $T_c^{(N)}$ between the metallic phase and the FISDW phases in presence of umklapp processes. The vertical lines are only guides for the eyes and do not necessarily correspond to the actual first-order transition lines between FISDW phases. (a) $g_3/g_2=0.03$. [g_3 and g_2 are the strengths of normal and umklapp processes, respectively (see Sec. II).] The shaded area corresponds to the Ribault phase $N=-2$. (b) $g_3/g_2=0.06$. Two negative phases, $N=-2$ and $N=-4$, are observed (shaded areas). The phase $N=-2$ splits into two subphases: helicoidal (dark shaded area) and sinusoidal (light shaded area).

C. Outline of the paper

In the next section, we introduce the effective Hamiltonian describing the FISDW phases. The partition function is written as a functional integral over a bosonic auxiliary field that describes spin fluctuations. In Sec. III, we perform a saddle-point approximation, thus recovering the mean-field results of Ref. 21. We obtain the mean-field propagators and the mean-field particle-hole susceptibilities. In Sec. IV, we derive the low-energy effective action of the SDW phase by taking into account fluctuations of the bosonic auxiliary field around its saddle-point value. We consider only "phase" fluctuations, i.e., sliding and spin-wave collective modes. We do not study amplitude collective modes, which are gapped and do not couple to phase fluctuations in the long-wavelength limit.[1]

We find two sliding modes: a (gapless) Goldstone mode corresponding to a sliding of the two SDW's in opposite directions (out-of-phase oscillations), and a gapped mode corresponding to in-phase oscillations (Sec. V). The real part of the conductivity exhibits two peaks, which reflect the presence of two sliding modes. The low-energy mode carries only a fraction of the total spectral weight $\omega_p^2/4$ (ω_p is the plasma frequency), which is determined by the ratio of the amplitudes of the two SDW's (Sec. VI).

The spin-wave modes are studied in Sec. VII. There is a Goldstone mode corresponding to in-phase oscillations of the two SDW's, and a gapped mode corresponding to out-of-phase oscillations. The spectral function $\mathrm{Im}\,\chi^{\rm ret}$ is computed in Sec. VIII ($\chi^{\rm ret}$ is the retarded transverse spin-spin correlation function). Both gapped modes are found to lie above the mean-field gap in the case of the Bechgaard salts.

In Sec. IX, we study the collective modes in the helicoidal phase where the two SDW's are circularly polarized. For a helicoidal structure, one cannot distinguish between a uniform spin rotation and a global translation, so that there are only two phase modes. The Goldstone mode does not contribute to the conductivity and is therefore a pure spin-wave mode. Thus all the spectral weight in the conductivity $\sigma(\omega)$ is pushed up above the mean-field gap. Both modes contribute to the spin-spin correlation function.

It should be noted that the long-wavelength modes are not the only modes of interest in the FISDW phases. There also exist magnetorotons at finite wave vectors ($q_x = G, 2G, \dots$).[25,26] These modes are not considered in this paper.

In the following, we take $\hbar = k_B = 1$.

II. MODEL AND EFFECTIVE HAMILTONIAN

In the vicinity of the Fermi energy, the electron dispersion law in the Bechgaard salts is approximated as

$$E(k_x,k_y)=v_F(|k_x|-k_F)+t_\perp(k_y b), \tag{2.1}$$

where k_x and k_y are the electron momenta along and across the one-dimensional chains of TMTSF. In Eq. (2.1), the longitudinal electron dispersion is linearized in k_x in the vicinity of the two one-dimensional Fermi points $\pm k_F$, and $v_F = 2at_a \sin(k_F a)$ is the corresponding Fermi velocity (t_a being the transfer integral and a the lattice spacing along the chains). Whenever necessary, we will impose an ultraviolet energy cutoff E_0 to simulate a finite bandwidth. The function $t_\perp(u)$, which describes the propagation in the transverse direction, is periodic: $t_\perp(u)=t_\perp(u+2\pi)$. It can be expanded in Fourier series as

$$t_\perp(u)=-2t_b\cos(u)-2t_{2b}\cos(2u) -2t_{3b}\cos(3u)-2t_{4b}\cos(4u)\cdots. \tag{2.2}$$

If we retain only the first harmonic (t_b), we obtain a Fermi surface with perfect nesting at $(2k_F,\pi/b)$. The other harmonics $t_{2b}, t_{3b}, \dots \ll t_b$ generate deviations from perfect nesting. They have been introduced in order to keep a realistic description of the Fermi surface despite the linearization around $\pm k_F$. In the following, we shall retain only t_b, t_{2b}, and t_{4b}. t_{3b} does not play an important role and can be discarded.[21] We do not consider the electron dispersion in the z direction, because it is not important in the following (its main effect is to introduce a 3D threshold field below which the FISDW cascade is suppressed[5]).

The effect of the magnetic field H along the z direction is taken into account via the Peierls substitution $\mathbf{k}\to -i\boldsymbol{\nabla} - e\mathbf{A}$. (The charge e is positive since the actual carriers are holes.) In the gauge $\mathbf{A}=(0,Hx,0)$ we obtain the noninteracting Hamiltonian

$$\mathcal{H}_0=\sum_{\alpha,\sigma}\int d^2r\,\hat\psi^\dagger_{\alpha\sigma}(\mathbf{r})[v_F(-i\alpha\partial_x-k_F) +t_\perp(-ib\partial_y-Gx)+\sigma h]\hat\psi_{\alpha\sigma}(\mathbf{r}). \tag{2.3}$$

Here $\hat\psi_{\alpha\sigma}(\mathbf{r})$ is a fermionic operator for right ($\alpha=+$) and left ($\alpha=-$) moving particles. $\sigma=+$ ($-$) for up (down) spin. We use the notation $\mathbf{r}=(x,mb)$ (m integer) and $\int d^2r = b\sum_m\int dx$. $G=eHb$ is a magnetic wave vector and $h=\mu_B H$ is the Zeeman energy (we assume the electron gyromagnetic factor to be equal to 2). Diagonalizing the Hamiltonian (2.3) we obtain the eigenstates and eigenenergies,

$$\phi^\alpha_{\mathbf{k}}(\mathbf{r})=\frac{1}{\sqrt{L_xL_y}}e^{i\mathbf{k}\cdot\mathbf{r}+i(\alpha/\omega_c)T_\perp(k_yb-Gx)},$$

$$\epsilon_{\alpha\sigma}(\mathbf{k})\equiv\epsilon_{\alpha\sigma}(k_x)=v_F(\alpha k_x-k_F)+\sigma h, \tag{2.4}$$

where L_xL_y is the area of the system, $\omega_c=v_FG$, and

$$T_\perp(u)=\int_0^u du'\,t_\perp(u'). \tag{2.5}$$

In the chosen gauge the energy depends only on k_x, i.e., the dispersion law is one dimensional. This reflects the localization of the electron motion in the transverse direction, which is at the origin of the QNM (see Ref. 21 for a further discussion). Note that, contrary to k_y, the quantum number k_x is not the momentum since the operator ∂_x does not commute with the Hamiltonian.

The interacting part of the Hamiltonian contains two terms corresponding to forward (g_2) and umklapp (g_3) scattering:

$$\mathcal{H}_{\rm int}=\frac{g_2}{2}\sum_{\alpha,\sigma,\sigma'}\int d^2r\,\hat\psi^\dagger_{\alpha\sigma}(\mathbf{r})\hat\psi^\dagger_{\bar\alpha\sigma'}(\mathbf{r})\hat\psi_{\bar\alpha\sigma'}(\mathbf{r})\hat\psi_{\alpha\sigma}(\mathbf{r}) +\frac{g_3}{2}\sum_{\alpha,\sigma}\int d^2r\,e^{-i\alpha 4k_Fx}\hat\psi^\dagger_{\alpha\sigma}(\mathbf{r})\hat\psi^\dagger_{\alpha\bar\sigma}(\mathbf{r})\hat\psi_{\bar\alpha\bar\sigma}(\mathbf{r})\hat\psi_{\bar\alpha\sigma}(\mathbf{r}), \tag{2.6}$$

where $\bar\alpha=-\alpha$ and $\bar\sigma=-\sigma$. For repulsive interaction $g_2,g_3\geq 0$. We shall assume that umklapp scattering is "weak": $g_3<g_2$. We do not consider backward scattering (g_1) since it does not play an important role in the following. The interaction strength is best parametrized by dimensionless coupling constants $\tilde g_2=g_2N(0)$ and $\tilde g_3=g_3N(0)$, where $N(0)=1/\pi v_F b$ is the density of states per spin.

In a mean-field theory, the cutoff E_0 is of the order of the bandwidth t_a. It is well known, however, that mean-field theory completely neglects fluctuations and cannot be directly applied to quasi-1D systems where the physics, at least at high temperature, is expected to be one dimensional. The general wisdom is that there is at low temperature a dimensional crossover from a 1D to a 2D (or 3D) regime.[27,28] In Bechgaard salts (at ambient pressure), this dimensional crossover occurs via coherent single-particle interchain hopping. Although the experimental assignment of the crossover temperature is still under debate (see, for instance, Refs. 29–31), the success of the QNM in explaining the phase diagram of the compounds $(TMTSF)_2PF_6$ and $(TMTSF)_2ClO_4$ provides clear evidence of the relevance of the Fermi surface at low temperature. In the 3D regime, a mean-field theory is justified provided that the parameters of the theory are understood as effective parameters (renormalized by 1D fluctuations).[27] E_0 is then smaller than the bare bandwidth ($\sim t_a$) and corresponds to the renormalized transverse bandwidth. For the same reason, the interaction amplitudes g_2 and g_3 can be different from their bare values. The Hamiltonian [Eqs. (2.3) and (2.6)] should therefore be understood as an effective low-energy Hamiltonian.

Since collective modes are best studied within a functional integral formalism, we write the partition function as $Z=\int\mathcal{D}\psi^*\mathcal{D}\psi e^{-S_0-S_{\rm int}}$ where $\psi^{(*)}$ is an anticommuting Grassmann variable. The actions S_0 and $S_{\rm int}$ are given by

$$S_0=\int d\tau\left\{\sum_{\alpha,\sigma}\int d^2r\,\psi^*_{\alpha\sigma}(\mathbf{r},\tau)\partial_\tau\psi_{\alpha\sigma}(\mathbf{r},\tau)+\mathcal{H}_0[\psi^*,\psi]\right\},$$

$$S_{\rm int}=-\frac{1}{2}\sum_{\alpha,\alpha',\sigma}\int d^2r\,d\tau\,O^*_{\alpha\sigma}(\mathbf{r},\tau)\hat{g}_{\alpha\alpha'}(\mathbf{r})O_{\alpha'\sigma}(\mathbf{r},\tau), \tag{2.7}$$

where the imaginary time τ varies between 0 and $\beta=1/T$. We have introduced the composite field

$$O_{\alpha\sigma}(\mathbf{r},\tau)=\psi^*_{\bar{\alpha}\sigma}(\mathbf{r},\tau)\psi_{\alpha\sigma}(\mathbf{r},\tau) \tag{2.8}$$

and the matrix

$$\hat{g}(\mathbf{r})=\begin{pmatrix} g_2 & g_3e^{i4k_Fx} \\ g_3e^{-i4k_Fx} & g_2\end{pmatrix}. \tag{2.9}$$

Introducing a complex auxiliary field $\Delta_{\alpha\sigma}(\mathbf{r},\tau)$, we decouple $S_{\rm int}$ by means of a Hubbard-Stratonovich transformation. This leads to the action

$$S=S_0+\int d^2r\,d\tau\,\Delta^\dagger_\uparrow(\mathbf{r},\tau)\hat{g}(\mathbf{r})\Delta_\uparrow(\mathbf{r},\tau)-\sum_\sigma\int d^2r\,d\tau\,\Delta^\dagger_\sigma(\mathbf{r},\tau)\hat{g}(\mathbf{r})O_\sigma(\mathbf{r},\tau). \tag{2.10}$$

We use the spinor notation $\Delta_\sigma=(\Delta_{+\sigma},\Delta_{-\sigma})^T$ and $O_\sigma=(O_{+\sigma},O_{-\sigma})^T$. Since $\Delta_{\alpha\sigma}$ couples to the field $O_{\alpha\sigma}(\mathbf{r},\tau)=O^*_{\bar{\alpha}\sigma}(\mathbf{r},\tau)$, it satisfies the constraint $\Delta_{\alpha\sigma}(\mathbf{r},\tau)=\Delta^*_{\bar{\alpha}\sigma}(\mathbf{r},\tau)$. Note that the action (2.10) maintains spin-rotation invariance around the magnetic field axis. In the FISDW phase, because of the Zeeman term, the magnetization is perpendicular to the magnetic field axis, so that the only spin-wave mode corresponds to rotation around the z axis. In zero magnetic field, a different approach should be used in order to maintain the full SU(2) spin symmetry (see Ref. 2).

III. MEAN-FIELD THEORY

In this section we look for a mean-field solution corresponding to a phase with two sinusoidal (i.e., linearly polarized) SDW's:

$$\langle S_x(\mathbf{r})\rangle=\sum_{p=\pm}M_{pN}\cos(\phi_{pN})\cos(\mathbf{Q}_{pN}\cdot\mathbf{r}+\theta_{pN}),$$

$$\langle S_y(\mathbf{r})\rangle=\sum_{p=\pm}M_{pN}\sin(\phi_{pN})\cos(\mathbf{Q}_{pN}\cdot\mathbf{r}+\theta_{pN}),$$

$$\langle S_z(\mathbf{r})\rangle=0, \tag{3.1}$$

where $S_\nu(\mathbf{r})=\sum_{\alpha\sigma\sigma'}\psi^*_{\bar{\alpha}\sigma}(\mathbf{r})\tau^\nu_{\sigma\sigma'}\psi_{\alpha\sigma'}(\mathbf{r})$ is the spin-density operator and τ^ν ($\nu=x,y,z$) the Pauli matrices. Because of the Zeeman coupling with the magnetic field, the SDW's are polarized in the (x,y) plane. The variable ϕ_{pN} determines the polarization axis, while θ_{pN} gives the position of the SDW's with respect to the underlying crystal lattice.

We assume that the external parameters (magnetic field, pressure, etc.) are such that the system is in the Ribault phase, characterized by a negative QHE and the coexistence of two SDW's with comparable amplitudes. We choose the sign of N such that N refers to the SDW with the largest amplitude ($M_N\geqslant M_{\bar{N}}$). (N is even and negative in the Ribault phase.)

The mean-field solution corresponds to a saddle-point approximation with a static auxiliary field[32]

$$\Delta_{\alpha\sigma}(\mathbf{r})=\langle O_{\alpha\sigma}(\mathbf{r},\tau)\rangle=\sum_p\Delta^{(pN)}_{\alpha\sigma}e^{i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}}. \tag{3.2}$$

The relation between $\Delta_{\alpha\sigma}$ and $\langle O_{\alpha\sigma}\rangle$ results from the stationarity condition of the saddle-point action [see Eq. (3.4) below]. Because of the constraint $\Delta^*_{\alpha\sigma}(\mathbf{r})=\Delta_{\bar{\alpha}\sigma}(\mathbf{r})$, the order parameters satisfy $\Delta^{(pN)*}_{\alpha\sigma}=\Delta^{(pN)}_{\bar{\alpha}\sigma}$. Among the eight complex order parameters, only four are therefore independent and sufficient to characterize the SDW phase. The order parameters $\Delta^{(pN)}_{\alpha\sigma}=|\Delta^{(pN)}_{\alpha\sigma}|e^{i\varphi^{(pN)}_{\alpha\sigma}}$ are related to the spin-density modulation (3.1) by

$$|\Delta^{(pN)}_{\alpha\sigma}|=\frac{M_{pN}}{4},$$

$$\theta_{pN}=\tfrac{1}{2}(\varphi^{(pN)}_{+\uparrow}-\varphi^{(pN)}_{-\uparrow}),$$

$$\phi_{pN}=-\tfrac{1}{2}(\varphi^{(pN)}_{+\uparrow}+\varphi^{(pN)}_{-\uparrow}). \tag{3.3}$$

Since by convention $M_N\geqslant M_{\bar{N}}$, we have $|\Delta^{(N)}_{\alpha\sigma}|\geqslant|\Delta^{(\bar{N})}_{\alpha\sigma}|$. Note that the condition $\Delta^{(pN)*}_{\alpha\sigma}=\Delta^{(pN)}_{\bar{\alpha}\sigma}$ imposes $\varphi^{(pN)}_{\bar{\alpha}\sigma}=-\varphi^{(pN)}_{\alpha\sigma}$.

The mean-field action is given by

$$S_{\rm MF}=\beta\int d^2r\Delta_\uparrow^\dagger(\mathbf{r})\hat{g}(\mathbf{r})\Delta_\uparrow(\mathbf{r})-\sum_\alpha \int d^2r d\tau\int d^2r' d\tau' (\psi^*_{\alpha\uparrow}(\mathbf{r},\tau),\psi^*_{\bar\alpha\downarrow}(\mathbf{r},\tau))$$
$$\times\begin{pmatrix} G^{(0)-1}_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau') & \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau')\tilde\Delta_{\alpha\uparrow}(\mathbf{r}) \\ \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau')\tilde\Delta^*_{\alpha\uparrow}(\mathbf{r}) & G^{(0)-1}_{\bar\alpha\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau')\end{pmatrix}\begin{pmatrix}\psi_{\alpha\uparrow}(\mathbf{r}',\tau') \\ \psi_{\bar\alpha\downarrow}(\mathbf{r}',\tau')\end{pmatrix}. \quad (3.4)$$

Here $G^{(0)}_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau')$ is the Green's function in the absence of interaction ($g_2=g_3=0$). The effective potential $\tilde\Delta_{\alpha\sigma}(\mathbf{r})$ acting on the electrons is defined by

$$\tilde\Delta_{\alpha\sigma}(\mathbf{r})=g_2\Delta_{\alpha\sigma}(\mathbf{r})+g_3 e^{i\alpha 4k_F x}\Delta_{\bar\alpha\sigma}(\mathbf{r}), \quad (3.5)$$

i.e., $\tilde\Delta_\sigma(\mathbf{r})=\hat g(\mathbf{r})\Delta_\sigma(\mathbf{r})$. Rewriting the action in the basis of the eigenstates $\phi^\alpha_\mathbf{k}$ of $\mathcal{H}_0$ [Eq. (2.4)], we obtain

$$S_{\rm MF}=\beta\int d^2r\Delta_\uparrow^\dagger(\mathbf{r})\hat g(\mathbf{r})\Delta_\uparrow(\mathbf{r})$$
$$-\sum_{\alpha,\mathbf{k},\mathbf{k}',\omega}(\psi^*_{\alpha\uparrow}(\mathbf{k},\omega),\psi^*_{\bar\alpha\downarrow}(\mathbf{k},\omega))$$
$$\times\begin{pmatrix}\delta_{\mathbf{k},\mathbf{k}'}[i\omega-\epsilon_{\alpha\uparrow}(k_x)] & \tilde\Delta_{\alpha\uparrow}(\mathbf{k},\mathbf{k}') \\ \tilde\Delta_{\bar\alpha\downarrow}(\mathbf{k},\mathbf{k}') & \delta_{\mathbf{k},\mathbf{k}'}[i\omega-\epsilon_{\bar\alpha\downarrow}(k_x)]\end{pmatrix}$$
$$\times\begin{pmatrix}\psi_{\alpha\uparrow}(\mathbf{k}',\omega) \\ \psi_{\bar\alpha\downarrow}(\mathbf{k}',\omega)\end{pmatrix}. \quad (3.6)$$

We denote by $\omega=2\pi T(n+1/2)$ (n integer) fermionic Matsubara frequencies. We emphasize that here $\mathbf{k}$ and $\mathbf{k}'$ refer to the quantum numbers of the eigenstates $\phi^\alpha_\mathbf{k}$ of $\mathcal{H}_0$. We have introduced the (particle-hole) pairing amplitudes

$$\tilde\Delta_{\alpha\sigma}(\mathbf{k},\mathbf{k}')=\int d^2r\,\phi^{\alpha *}_\mathbf{k}(\mathbf{r})\phi^{\bar\alpha}_{\mathbf{k}'}(\mathbf{r})\tilde\Delta_{\alpha\sigma}(\mathbf{r})$$
$$=\delta_{k'_y,k_y-\pi/b}\sum_p (g_2\Delta^{(pN)}_{\alpha\sigma}+g_3\Delta^{(\overline{pN})}_{\alpha\sigma})$$
$$\times\sum_{n=-\infty}^{\infty} I_n e^{i\alpha n(k_y b-\pi/2)}\delta_{k'_x,k_x-\alpha Q_x^{(pN)}+\alpha nG}, \quad (3.7)$$

where $I_n\equiv I_n(q_y=\pi/b)$. The coefficients $I_n(q_y)$, defined by

$$I_n(q_y)=\int_0^{2\pi}\frac{du}{2\pi}e^{inu+(i/\omega_c)[T_\perp(u+q_y b/2)+T_\perp(u-q_y b/2)]}, \quad (3.8)$$

are well known in the QNM. They depend on the transverse dispersion law $t_\perp(k_y b)$ and measure the degree of perfect nesting of the Fermi surface. For a perfect nesting at wave vector $(2k_F,\pi/b)$, one would have $I_n(\pi/b)=\delta_{n,0}$, while in the generic situation (imperfect nesting) $|I_n(q_y)|<1$.[33] The convention $M_{\bar N}\leqslant M_N$ implies $|I_{\bar N}|\leqslant|I_N|$. Note that k_x is coupled not only to $k_x-\alpha Q_x^{(pN)}$ but also to $k_x-\alpha Q_x^{(pN)}+\alpha nG$ [Eq. (3.7)]. This reflects the fact that k_x is not the momentum in the gauge $\mathbf{A}=(0,Hx,0)$.

From now on, we shall use the quantum limit approximation (QLA) (also known as the single-gap approximation[5]). This approximation, which is valid when ω_c is much larger than the temperature and the SDW gap, consists in retaining only the most singular (particle-hole) pairing channels. In the metallic phase, these channels present the logarithmic singularity $\sim\ln(E_0/T)$ (which is characteristic of a SDW system with perfect nesting). This singularity results from pairing between electron and hole states of the same energy and is responsible for the opening of a gap at the Fermi level. While the QLA strongly underestimates the transition temperature and the value of the gap at low temperature, and also neglects gaps opening above and below the Fermi level, it is an excellent approximation when calculating the collective modes (we shall come back to this point).[25]

Thus, in the QLA, $\tilde\Delta_{\alpha\sigma}(\mathbf{k},\mathbf{k}')$ is not zero only if $\epsilon_{\alpha\sigma}(k_x)=-\epsilon_{\overline{\alpha\sigma}}(k'_x)$, which requires $k'_x=k_x-\alpha 2k_F$. Only the term with $n=pN$ survives in Eq. (3.7), which yields

$$\tilde\Delta_{\alpha\sigma}(\mathbf{k},\mathbf{k}')=\delta_{\mathbf{k}',\mathbf{k}-\alpha\mathbf{Q}_0}\tilde\Delta_{\alpha\sigma}(k_y),$$
$$\tilde\Delta_{\alpha\sigma}(k_y)=\sum_p\tilde\Delta^{(pN)}_{\alpha\sigma}e^{i\alpha pN(k_y b-\pi/2)}, \quad (3.9)$$

where $\mathbf{Q}_0=(2k_F,\pi/b)$. We have introduced the new order parameters $\tilde\Delta^{(pN)}_{\alpha\sigma}$ related to $\Delta^{(pN)}_{\alpha\sigma}$ by

$$\tilde\Delta^{(pN)}_{\alpha\sigma}=I_{pN}(g_2\Delta^{(pN)}_{\alpha\sigma}+g_3\Delta^{(\overline{pN})}_{\alpha\sigma})=\tilde\Delta^{(\overline{pN})*}_{\alpha\sigma},$$
$$\Delta^{(pN)}_{\alpha\sigma}=\frac{g_2 I_{\overline{pN}}\tilde\Delta^{(pN)}_{\alpha\sigma}-g_3 I_{pN}\tilde\Delta^{(\overline{pN})}_{\alpha\sigma}}{(g_2^2-g_3^2)I_N I_{\bar N}}. \quad (3.10)$$

In the QLA, the mean-field action therefore reduces to a 2 $\times$2 matrix (as in BCS theory):

$$S_{\rm MF}=\beta\int d^2r\Delta_\uparrow^\dagger(\mathbf{r})\hat g(\mathbf{r})\Delta_\uparrow(\mathbf{r})$$
$$-\sum_{\alpha,\mathbf{k},\omega}(\psi^*_{\alpha\uparrow}(\mathbf{k},\omega),\psi^*_{\bar\alpha\downarrow}(\mathbf{k}-\alpha\mathbf{Q}_0,\omega))$$
$$\times\begin{pmatrix} i\omega-\epsilon_{\alpha\uparrow}(k_x) & \tilde\Delta_{\alpha\uparrow}(k_y) \\ \tilde\Delta^*_{\alpha\uparrow}(k_y) & i\omega+\epsilon_{\alpha\uparrow}(k_x)\end{pmatrix}$$
$$\times\begin{pmatrix}\psi_{\alpha\uparrow}(\mathbf{k},\omega) \\ \psi_{\bar\alpha\downarrow}(\mathbf{k}-\alpha\mathbf{Q}_0,\omega)\end{pmatrix}, \quad (3.11)$$

using $\epsilon^{-}_{\bar{\alpha}\downarrow}(k_x-\alpha 2k_F)=-\epsilon_{\alpha\uparrow}(k_x)$. It is remarkable that within the QLA the mean-field action can still be written as a 2×2 matrix. The presence of a second SDW changes the expression of the pairing amplitude (which becomes k_y dependent), but not the fact that the state $(\mathbf{k},-)$ couples only to $(\mathbf{k}+\mathbf{Q}_0,+)$. This is not true in zero magnetic field where the presence of a second SDW leads to complicated mean-field equations.[10]

A. Ground-state energy

The mean-field action being Gaussian, we can integrate out the fermion fields to obtain the ground-state condensation energy (per unit area), $\Delta E=-(T\ln Z)/L_xL_y-E_N$, where $T\to 0$ and $E_N=-N(0)E_0^2$ is the normal state energy:

$$\Delta E=\sum_\alpha\left[\sum_p\frac{\Delta^{(pN)*}_{\alpha\uparrow}\tilde{\Delta}^{(pN)}_{\alpha\uparrow}}{I_{pN}}+\frac{N(0)}{2}|\tilde{\Delta}^{(\bar{N})}_{\alpha\uparrow}|^2-\frac{N(0)}{2}\sum_p|\tilde{\Delta}^{(pN)}_{\alpha\uparrow}|^2\left(\frac{1}{2}+\ln\frac{2E_0}{|\tilde{\Delta}^{(N)}_{\alpha\uparrow}|}\right)\right]. \tag{3.12}$$

To obtain Eq. (3.12), we have explicitly taken into account the facts that $|\tilde{\Delta}^{(N)}_{\alpha\sigma}|\geqslant|\tilde{\Delta}^{(\bar{N})}_{\alpha\sigma}|$ and $|\Delta^{(N)}_{\alpha\sigma}|\geqslant|\Delta^{(\bar{N})}_{\alpha\sigma}|$. The latter inequality follows from $M_N\geqslant M_{\bar{N}}$, while the former is derived below Eq. (3.15). Using the gap equation $\partial\Delta E/\partial\tilde{\Delta}^{(pN)*}_{\alpha\uparrow}=0$, we can rewrite the condensation energy as

$$\Delta E=-\frac{N(0)}{4}\sum_{\alpha,p}|\tilde{\Delta}^{(pN)}_{\alpha\uparrow}|^2, \tag{3.13}$$

a form which is reminiscent of the BCS-like expression of a SDW system with perfect nesting $[\Delta E=-N(0)|\Delta|^2/2$ with Δ the SDW gap]. As stated above, the QLA amounts to taking into account only the ''perfect nesting component'' of the particle-hole pairing.

The requirement that ΔE is minimal with respect to variation of the order parameters leads to constraints on the phases of the complex order parameters $\tilde{\Delta}^{(pN)}_{\alpha\sigma}=|\tilde{\Delta}^{(pN)}_{\alpha\sigma}|e^{i\tilde{\varphi}^{(pN)}_{\alpha\sigma}}$:

$$\tilde{\varphi}^{(N)}_{\alpha\uparrow}-\tilde{\varphi}^{(\bar{N})}_{\alpha\uparrow}=\begin{cases}0 & \text{if } I_NI_{\bar{N}}>0\\ \pi & \text{if } I_NI_{\bar{N}}<0.\end{cases} \tag{3.14}$$

For $t_{4b}=0$, $I_{\bar{N}}=(-1)^{N/2}I_N$. When t_{4b} is finite but small, $\mathrm{sgn}(I_NI_{\bar{N}})=(-1)^{N/2}$. Using Eqs. (3.10) and (3.14), we then obtain

$$\frac{\Delta^{(\bar{N})}_{\alpha\uparrow}}{\Delta^{(N)}_{\alpha\uparrow}}=\frac{|\tilde{\Delta}^{(\bar{N})}_{\alpha\uparrow}/\tilde{\Delta}^{(N)}_{\alpha\uparrow}|-r|I_{\bar{N}}/I_N|}{|I_{\bar{N}}/I_N|-r|\tilde{\Delta}^{(\bar{N})}_{\alpha\uparrow}/\tilde{\Delta}^{(N)}_{\alpha\uparrow}|}. \tag{3.15}$$

Equation (3.15), together with $|\Delta^{(\bar{N})}_{\alpha\sigma}/\Delta^{(N)}_{\alpha\sigma}|=M_{\bar{N}}/M_N\leqslant 1$ and $|I_{\bar{N}}/I_N|\leqslant 1$, implies $|\tilde{\Delta}^{(\bar{N})}_{\alpha\sigma}/\tilde{\Delta}^{(N)}_{\alpha\sigma}|\leqslant 1$. It also shows that $\Delta^{(\bar{N})}_{\alpha\uparrow}/\Delta^{(N)}_{\alpha\uparrow}$ is a real number, so that the phases $\varphi^{(N)}_{\alpha\uparrow}$ and $\varphi^{(\bar{N})}_{\alpha\uparrow}$ are identical modulo π:

$$\varphi^{(N)}_{\alpha\uparrow}=\varphi^{(\bar{N})}_{\alpha\uparrow}\quad[\pi]. \tag{3.16}$$

From Eqs. (3.3) and (3.16), we then deduce

$$\phi_N=\phi_{\bar{N}},\quad \theta_N=-\theta_{\bar{N}}. \tag{3.17}$$

In the mean-field ground state, the two SDW's have the same polarization axis ($\phi_N=\phi_{\bar{N}}$). The condition $\theta_N=-\theta_{\bar{N}}$ means that the two SDW's can be displaced in opposite directions without changing the energy of the system. They can therefore be centered either on the lattice sites or on the bonds. For $\theta_N=\theta_{\bar{N}}=0$, both SDW's are centered on the sites, while for $\theta_N=-\theta_{\bar{N}}=\pi/2$, they are centered on the bonds. The condition $\theta_N+\theta_{\bar{N}}=0$ is related to the pinning that would occur for a commensurate SDW. Indeed, for a single SDW with wave vector $(2k_F,\pi/b)$, it becomes $2\theta=0$ where θ is the phase of the SDW.[1]

This degeneracy of the ground state results from rotational invariance around the z axis in spin space,[34] and translational invariance in real space. The latter holds in the FISDW states, since the SDW's are incommensurate with respect to the crystal lattice.[35] According to the mean-field analysis, and in agreement with general symmetry considerations, we therefore expect two (gapless) Goldstone modes: a spin-wave mode corresponding to a uniform rotation around the z axis of the common polarization axis, and a sliding mode corresponding to a displacement of the two SDW's in opposite directions.

Without loss of generality, we can choose the phases $\tilde{\varphi}^{(pN)}_{\alpha\sigma}$ and $\varphi^{(pN)}_{\alpha\sigma}$ equal to 0 or π. The order parameters $\tilde{\Delta}^{(pN)}_{\alpha\sigma}$ and $\Delta^{(pN)}_{\alpha\sigma}$ are then real (positive or negative) numbers. Furthermore, since the constraints obtained by minimizing the free energy relate $\tilde{\varphi}^{(N)}_{\alpha\sigma}$ $(\varphi^{(N)}_{\alpha\sigma})$ only to $\tilde{\varphi}^{(\bar{N})}_{\alpha\sigma}$ $(\varphi^{(\bar{N})}_{\alpha\sigma})$, we can also impose $\tilde{\varphi}^{(N)}_{\alpha\sigma}=\tilde{\varphi}^{(N)}_{\alpha\sigma}$ and $\varphi^{(N)}_{\alpha\sigma}=\varphi^{(N)}_{\alpha\sigma}$. With such a choice,

$$\tilde{\Delta}^{(pN)}_{\alpha\sigma}\equiv\tilde{\Delta}_{pN},\quad \Delta^{(pN)}_{\alpha\sigma}\equiv\Delta_{pN} \tag{3.18}$$

are real (positive or negative) numbers independent of α and σ. $\theta_{pN}=0$ and $\phi_N=\phi_{\bar{N}}=0$ or π. The SDW's are polarized along the x axis. Introducing the notations

$$r=\frac{g_3}{g_2},\quad \zeta=\frac{I_{\bar{N}}}{I_N},$$

$$\gamma=\frac{\Delta_{\bar{N}}}{\Delta_N},\quad \tilde{\gamma}=\frac{\tilde{\Delta}_{\bar{N}}}{\tilde{\Delta}_N}, \tag{3.19}$$

(with $|\gamma|=M_{\bar{N}}/M_N$), we deduce from Eq. (3.10)

$$\tilde{\gamma}=\zeta\frac{\gamma+r}{1+r\gamma},\quad \gamma=\frac{\tilde{\gamma}-r\zeta}{\zeta-r\tilde{\gamma}}. \tag{3.20}$$

Note that $\tilde{\gamma}$ and ζ have the same sign [see Eq. (3.14)], and $|\gamma|,|\tilde{\gamma}|,r,|\zeta|\leqslant 1$. In the Ribault phase, $|\gamma|\simeq|\tilde{\gamma}|$.[21] Both SDW's have the same amplitude when $|\zeta|=1$. In that case $\tilde{\gamma}=\zeta$ and $\gamma=1$.

B. Mean-field propagators

The normal and anomalous Green's functions are defined by

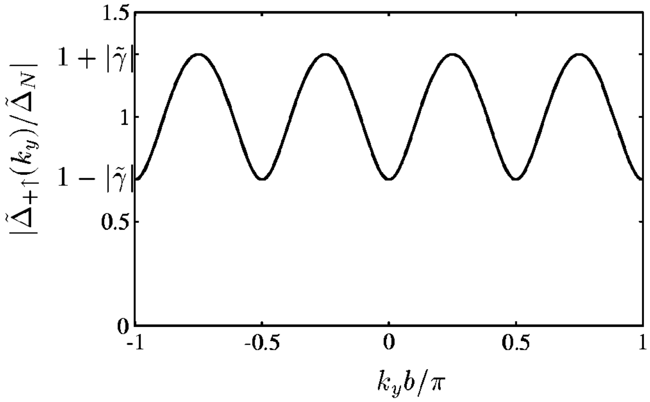

FIG. 3. Quasiparticle excitation gap $2|\tilde{\Delta}_{+\uparrow}(k_y)|$ in the phase $N=-2$ for $|\tilde{\gamma}|=1/3$.

$$\begin{aligned} G_{\alpha\sigma}(\mathbf{r},\mathbf{r}',\omega) &= -\langle \psi_{\alpha\sigma}(\mathbf{r},\omega)\psi^*_{\alpha\sigma}(\mathbf{r}',\omega)\rangle \\ &= \sum_{\mathbf{k},\mathbf{k}'} G_{\alpha\sigma}(\mathbf{k},\mathbf{k}',\omega)\phi^{\alpha}_{\mathbf{k}}(\mathbf{r})\phi^{\alpha *}_{\mathbf{k}'}(\mathbf{r}'), \\ F_{\alpha\sigma}(\mathbf{r},\mathbf{r}',\omega) &= -\langle \psi_{\alpha\sigma}(\mathbf{r},\omega)\psi^*_{\overline{\alpha\sigma}}(\mathbf{r}',\omega)\rangle \\ &= \sum_{\mathbf{k},\mathbf{k}'} F_{\alpha\sigma}(\mathbf{k},\mathbf{k}',\omega)\phi^{\alpha}_{\mathbf{k}}(\mathbf{r})\phi^{\bar{\alpha} *}_{\mathbf{k}'}(\mathbf{r}'). \end{aligned} \tag{3.21}$$

From Eq. (3.11), we deduce that

$$\begin{aligned} G_{\alpha\sigma}(\mathbf{k},\mathbf{k}',\omega) &= \delta_{\mathbf{k},\mathbf{k}'} G_{\alpha\sigma}(\mathbf{k},\omega), \\ F_{\alpha\sigma}(\mathbf{k},\mathbf{k}',\omega) &= \delta_{\mathbf{k}',\mathbf{k}-\alpha\mathbf{Q}_0} F_{\alpha\sigma}(\mathbf{k},\omega), \end{aligned} \tag{3.22}$$

with

$$\begin{aligned} G_{\alpha\sigma}(\mathbf{k},\omega) &= \frac{-i\omega-\epsilon_{\alpha\sigma}(k_x)}{\omega^2+\epsilon^2_{\alpha\sigma}(k_x)+|\tilde{\Delta}_{\alpha\sigma}(k_y)|^2}, \\ F_{\alpha\sigma}(\mathbf{k},\omega) &= \frac{\tilde{\Delta}_{\alpha\sigma}(k_y)}{\omega^2+\epsilon^2_{\alpha\sigma}(k_x)+|\tilde{\Delta}_{\alpha\sigma}(k_y)|^2}. \end{aligned} \tag{3.23}$$

The excitation energies are given by $\pm E_{\alpha\sigma}(\mathbf{k})$, where

$$E_{\alpha\sigma}(\mathbf{k}) = \sqrt{\epsilon^2_{\alpha\sigma}(k_x)+|\tilde{\Delta}_{\alpha\sigma}(k_y)|^2}. \tag{3.24}$$

Writing $\tilde{\Delta}^{(pN)}_{\alpha\uparrow} = |\tilde{\Delta}^{(pN)}_{\alpha\uparrow}| e^{i\tilde{\varphi}^{(pN)}_{\alpha\sigma}}$, we obtain (for N even)

$$\begin{aligned} |\tilde{\Delta}_{\alpha\sigma}(k_y)|^2 &= \sum_p |\tilde{\Delta}_{pN}|^2 + 2|\tilde{\Delta}_N\tilde{\Delta}_{\bar{N}}| \\ &\quad \times \cos(2\alpha N k_y b + \tilde{\varphi}^{(N)}_{\alpha\sigma} - \tilde{\varphi}^{(\bar{N})}_{\alpha\sigma}). \end{aligned} \tag{3.25}$$

The quasiparticle excitation gap $2|\tilde{\Delta}_{\alpha\sigma}(k_y)|$ depends on the transverse momentum k_y (Fig. 3). When both SDW's have the same amplitude, $|\tilde{\Delta}_N| = |\tilde{\Delta}_{\bar{N}}|$, the spectrum becomes gapless since $\tilde{\Delta}_{\alpha\sigma}(k_y)$ vanishes at some points on the Fermi surface.

We have shown in Ref. 21 that the occurrence of helicoidal phases prevents the spectrum from becoming gapless. Indeed, the stability of the sinusoidal phase requires

$$\frac{\min_{k_y}|\tilde{\Delta}_{\alpha\sigma}(k_y)|}{\max_{k_y}|\tilde{\Delta}_{\alpha\sigma}(k_y)|} = \frac{1-|\tilde{\gamma}|}{1+|\tilde{\gamma}|} \gtrsim 0.32, \tag{3.26}$$

i.e., $|\tilde{\gamma}| \lesssim 0.52$. Equation (3.26) is deduced from a Ginzburg-Landau expansion of the free energy (valid only close to the transition line) and may be slightly changed at low temperature.

In the helicoidal phase, there is no dispersion of the gap with respect to k_y (see Sec. IX). Since the spectrum becomes gapless for $|\tilde{\gamma}|=1$, it is natural that the system, above a certain value of $|\tilde{\gamma}|$, prefers to form helicoidal SDW's in order to lower the free energy by opening a large gap on the whole Fermi surface.

C. Mean-field susceptibilities

The results reported in this section are derived in detail in Appendix A. We study the mean-field susceptibilities (which will be useful in Sec. IV)

$$\begin{aligned} \bar{\chi}_{\alpha\sigma,\alpha'\sigma'}(\mathbf{r},\tau;\mathbf{r}',\tau') &= \langle O_{\alpha\sigma}(\mathbf{r},\tau) O^*_{\alpha'\sigma'}(\mathbf{r}',\tau')\rangle \\ &\quad - \langle O_{\alpha\sigma}(\mathbf{r},\tau)\rangle\langle O^*_{\alpha'\sigma'}(\mathbf{r}',\tau')\rangle, \end{aligned} \tag{3.27}$$

where the mean value $\langle\cdots\rangle$ is taken with the mean-field action $S_{\rm MF}$ [Eq. (3.11)]. Since the latter is Gaussian, we immediately deduce the only two nonvanishing components:

$$\begin{aligned} \bar{\chi}_{\alpha\sigma,\alpha\sigma}(\mathbf{r},\tau;\mathbf{r}',\tau') &= -G_{\alpha\sigma}(\mathbf{r},\tau;\mathbf{r}',\tau') G_{\overline{\alpha\sigma}}(\mathbf{r}',\tau';\mathbf{r},\tau), \\ \bar{\chi}_{\alpha\sigma,\overline{\alpha\sigma}}(\mathbf{r},\tau;\mathbf{r}',\tau') &= -F_{\alpha\sigma}(\mathbf{r},\tau;\mathbf{r}',\tau') F_{\alpha\sigma}(\mathbf{r}',\tau';\mathbf{r},\tau). \end{aligned} \tag{3.28}$$

At $T=0$ and for $q_y=0$, the Fourier-transformed susceptibilities are given by (Appendix A)

$$\begin{aligned} &\bar{\chi}_{+\uparrow,+\uparrow}(\mathbf{q}+\mathbf{Q}_{pN},\mathbf{q}+\mathbf{Q}_{pN},\omega_\nu) \\ &\quad = \frac{N(0)}{2} I^2_{pN}\left(\ln\frac{2E_0}{|\tilde{\Delta}_N|} - \frac{1}{2} - \frac{\omega^2_\nu + v^2_F q^2_x}{6(1-\tilde{\gamma}^2)\tilde{\Delta}^2_N}\right), \\ &\bar{\chi}_{+\uparrow,+\uparrow}(\mathbf{q}+\mathbf{Q}_{pN},\mathbf{q}+\mathbf{Q}_{\bar{p}N},\omega_\nu) \\ &\quad = -\frac{N(0)}{4} I_N I_{\bar{N}}\tilde{\gamma}\left(1 - \frac{\omega^2_\nu + v^2_F q^2_x}{3(1-\tilde{\gamma}^2)\tilde{\Delta}^2_N}\right), \\ &\bar{\chi}_{+\uparrow,-\downarrow}(\mathbf{q}+\mathbf{Q}_{pN},\mathbf{q}-\mathbf{Q}_{pN},\omega_\nu) \\ &\quad = -\delta_{p,+}\frac{N(0)}{4} I^2_N(1-\tilde{\gamma}^2)\left(1 - \frac{\omega^2_\nu + v^2_F q^2_x}{6(1-\tilde{\gamma}^2)\tilde{\Delta}^2_N}\right), \\ &\bar{\chi}_{+\uparrow,-\downarrow}(\mathbf{q}+\mathbf{Q}_{pN},\mathbf{q}-\mathbf{Q}_{\bar{p}N},\omega_\nu) = -\frac{N(0)}{4} I_N I_{\bar{N}}\tilde{\gamma}, \end{aligned} \tag{3.29}$$

where $\omega_\nu = 2\pi T \nu$ (ν integer) is a bosonic Matsubara frequency. Equations (3.29) are valid in the low-energy limit $v_F^2 q_x^2, \omega_\nu^2 \ll \tilde{\Delta}_N^2(1-\tilde{\gamma}^2)$. We have used the QLA (see Appendix A), whose validity is discussed in detail in Ref. 25. Although the QLA does not predict accurately the value of the mean-field gaps $\tilde{\Delta}_{pN}$ and Δ_{pN}, it is an excellent approximation of the low-energy properties (such as the long-wavelength collective modes) once the values of $\tilde{\Delta}_{pN}$ and Δ_{pN} are known. Since the absolute value of the latter do not play an important role for our purpose, the QLA turns out to be a perfect mean to compute the collective modes.

IV. LOW-ENERGY EFFECTIVE ACTION

In this section we derive the low-energy effective action determining the sliding and spin-wave modes. We do not consider amplitude modes, which are gapped and do not couple to ''phase'' fluctuations in the long-wavelength limit. We consider only longitudinal fluctuations ($q_y=0$) (i.e., those with the wave vector parallel to the chains). These fluctuations are of particular interest since they couple to an electric field applied along the chains (which is a common experimental situation): see Sec. VI on the optical conductivity. Including a finite q_y would allow one to obtain the effective mass of the collective modes in the transverse direction. Such a calculation is, however, much more involved and will not be attempted here.

The low-energy effective action is derived by studying fluctuations of the auxiliary field $\Delta_{\alpha\sigma}(\mathbf{r},\tau)$ around its saddle-point value $\Delta_{\alpha\sigma}(\mathbf{r})$. We write $\Delta_{\alpha\sigma}(\mathbf{r},\tau)=\Delta_{\alpha\sigma}(\mathbf{r}) + \eta_{\alpha\sigma}(\mathbf{r},\tau)$ and calculate the effective action to quadratic order in the fluctuating field η. The fermionic action (2.10) can be rewritten as

$$S[\psi^*,\psi]=S_{\rm MF}[\psi^*,\psi]-\sum_\sigma \int d^2r\,d\tau\,\eta^\dagger_\sigma(\mathbf{r},\tau)\tilde{O}_\sigma(\mathbf{r},\tau) + \int d^2r\,d\tau\{\eta^\dagger_\uparrow(\mathbf{r},\tau)\hat{g}(\mathbf{r})\,\eta_\uparrow(\mathbf{r},\tau) + [\eta^\dagger_\uparrow(\mathbf{r},\tau)\tilde{\Delta}_\uparrow(\mathbf{r})+\text{c.c.}]\}. \tag{4.1}$$

Integrating out the fermions, we obtain to quadratic order in η the effective action

$$S[\eta^*,\eta]=\frac{1}{2}\int d^2r\,d\tau\int d^2r'\,d\tau' \sum_{\alpha,\alpha',\sigma,\sigma'} \eta^*_{\alpha\sigma}(\mathbf{r},\tau) \times[\delta_{\sigma,\sigma'}\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau')\hat{g}_{\alpha\alpha'}(\mathbf{r}) - \tilde{\chi}_{\alpha\sigma,\alpha'\sigma'}(\mathbf{r},\tau;\mathbf{r}',\tau')]\,\eta_{\alpha'\sigma'}(\mathbf{r}',\tau'), \tag{4.2}$$

where the susceptibility $\tilde{\chi}$ is defined by Eq. (A13). There is no linear term since we expand around the saddle point. Considering phase fluctuations only,

$$\Delta_{\alpha\sigma}(\mathbf{r},\tau)=\sum_p \Delta_{pN} e^{i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}+i\varphi^{(pN)}_{\alpha\sigma}(\mathbf{r},\tau)}, \tag{4.3}$$

we have

$$\eta_{\alpha\sigma}(\mathbf{r},\tau)=i\sum_p \Delta_{pN}e^{i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}}\varphi^{(pN)}_{\alpha\sigma}(\mathbf{r},\tau) \tag{4.4}$$

to lowest order in the (real) phase field $\varphi^{(pN)}_{\alpha\sigma}(\mathbf{r},\tau)$. Note that the property $\Delta^*_{\alpha\sigma}(\mathbf{r},\tau)=\Delta_{\overline{\alpha\sigma}}(\mathbf{r},\tau)$ imposes $\varphi^{(pN)}_{\alpha\sigma}(\mathbf{r},\tau)= -\varphi^{(pN)}_{\overline{\alpha\sigma}}(\mathbf{r},\tau)$. Using this latter condition, we deduce from Eq. (4.2) the effective action for the phase:

$$S[\varphi]=\sum_{p,p'}\Delta_{pN}\Delta_{p'N}\sum_{\alpha,\alpha',\tilde{q}}\varphi^{(pN)}_{\alpha\uparrow}(-\tilde{q})\varphi^{(p'N)}_{\alpha'\uparrow}(\tilde{q}) \times[g_2\delta_{\alpha,\alpha'}\delta_{p,p'}+g_3\delta_{\alpha,\bar{\alpha}'}\delta_{p,\bar{p}'}+A^{pp'}_{\alpha\alpha'}(\tilde{q})], \tag{4.5}$$

where $\tilde{q}=(q_x,\omega_\nu)$, and

$$A^{pp'}_{\alpha\alpha'}(\tilde{q})=-\tilde{\chi}_{\alpha\uparrow,\alpha'\uparrow}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}+\alpha'\mathbf{Q}_{p'N},\omega_\nu) +\tilde{\chi}_{\alpha\uparrow,\bar{\alpha}'\downarrow}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}-\alpha'\mathbf{Q}_{p'N},\omega_\nu). \tag{4.6}$$

In Eq. (4.6) $\mathbf{q}=(q_x,q_y=0)$. Using Eq. (A15), and introducing charge and spin variables as in the mean-field theory [see Eq. (3.3)],

$$\theta_{pN}(\mathbf{r},\tau)=\tfrac{1}{2}[\varphi^{(pN)}_{+\uparrow}(\mathbf{r},\tau)-\varphi^{(pN)}_{-\uparrow}(\mathbf{r},\tau)],$$
$$\phi_{pN}(\mathbf{r},\tau)=-\tfrac{1}{2}[\varphi^{(pN)}_{+\uparrow}(\mathbf{r},\tau)+\varphi^{(pN)}_{-\uparrow}(\mathbf{r},\tau)], \tag{4.7}$$

we obtain a decoupling of the sliding and spin-wave modes, which are determined by the actions

$$S[\theta]=\frac{1}{2}\sum_{\tilde{q}}(\theta_N(-\tilde{q}),\theta_{\bar{N}}(-\tilde{q}))\mathcal{D}^{-1}_{\rm ch}(\tilde{q})\begin{pmatrix}\theta_N(\tilde{q})\\ \theta_{\bar{N}}(\tilde{q})\end{pmatrix}, \tag{4.8}$$

$$S[\phi]=\frac{1}{2}\sum_{\tilde{q}}(\phi_N(-\tilde{q}),\phi_{\bar{N}}(-\tilde{q}))\mathcal{D}^{-1}_{\rm sp}(\tilde{q})\begin{pmatrix}\phi_N(\tilde{q})\\ \phi_{\bar{N}}(\tilde{q})\end{pmatrix}, \tag{4.9}$$

with

$$\mathcal{D}^{-1}_{\rm ch}(\tilde{q})=4g_2\Delta_N^2\begin{pmatrix}1-c_{++}-r^2c_{--}+2rc_{+-} & -\gamma r(1-c_{++}-c_{--})-\gamma(1+r^2)c_{+-}\\ -\gamma r(1-c_{++}-c_{--})-\gamma(1+r^2)c_{+-} & \gamma^2(1-c_{--}-r^2c_{++})+2\gamma^2rc_{+-}\end{pmatrix}, \tag{4.10}$$

$$\mathcal{D}^{-1}_{\rm sp}(\tilde{q})=4g_2\Delta_N^2\begin{pmatrix}1-c_{++}-r^2c_{--}-2rc_{+-} & \gamma r(1-c_{++}-c_{--})-\gamma(1+r^2)c_{+-}\\ \gamma r(1-c_{++}-c_{--})-\gamma(1+r^2)c_{+-} & \gamma^2(1-c_{--}-r^2c_{++})-2\gamma^2rc_{+-}\end{pmatrix}. \tag{4.11}$$

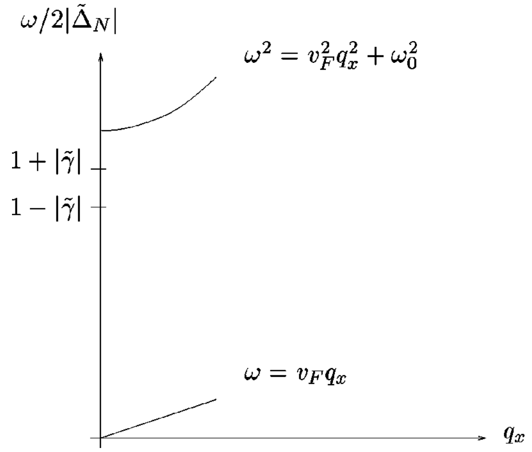

FIG. 4. Dispersion laws of the two sliding modes in the Ribault phase. The gapped mode is generally expected to lie above the quasiparticle excitation gap. The latter varies between $2|\tilde{\Delta}_N|(1-|\tilde{\gamma}|)$ and $2|\tilde{\Delta}_N|(1+|\tilde{\gamma}|)$ (Fig. 3).

The coefficients $c_{pp'} \equiv c_{pp'}(\tilde{q})$ are linear combinations of the mean-field susceptibilities $\underline{\chi}$ and are defined in Appendix A.

The effective actions [Eqs. (4.8) and (4.9)] have been obtained by expanding the auxiliary field with respect to the phase fluctuations $\varphi_{\alpha\sigma}^{(pN)}(\mathbf{r},\tau)$ [Eq. (4.4)]. While this procedure turns out to be correct at zero temperature, an alternative and more rigorous approach consists in performing a chiral rotation of the fermion fields, which allows one to directly expand with respect to $\partial_\tau \varphi_{\alpha\sigma}^{(pN)}(\mathbf{r},\tau)$ and $\nabla \varphi_{\alpha\sigma}^{(pN)}(\mathbf{r},\tau)$.[2,36] The drawback of this approach is that one has to calculate the nontrivial Jacobian of the chiral transformation (the so-called chiral anomaly).

V. SLIDING MODES

The dispersion of the sliding modes is obtained from $\det \mathcal{D}_{\rm ch}^{-1}(\tilde{q})=0$. From Eq. (4.10), we deduce

$$\det \mathcal{D}_{\rm ch}^{-1}(\tilde{q}) = 16 g_2^2 \Delta_N^4 \gamma^2 (1-r^2)[(1-c_{++})(1-c_{--}) - r^2 c_{++}c_{--} + 2rc_{+-} - (1-r^2)c_{+-}^2]. \tag{5.1}$$

The gap equation (A12) shows that $\det \mathcal{D}_{\rm ch}^{-1}(\tilde{q}=0)=0$, which indicates that there is a gapless mode with vanishing energy in the limit of long wavelength ($q_x \to 0$). Equation (5.1) simplifies into

$$\delta c_{++}\frac{r\zeta}{\tilde{\gamma}} + \delta c_{--}\frac{r\tilde{\gamma}}{\zeta} - 2r\,\delta c_{+-} - (1-r^2)(\delta c_{++}\delta c_{--} - \delta c_{+-}^2) = 0. \tag{5.2}$$

A. Goldstone mode

Equation (5.2) admits the solution $\omega_\nu^2 + v_F^2 q_x^2 = 0$. After analytical continuation to real frequencies ($i\omega_\nu \to \omega + i0^+$), we obtain a mode with a linear dispersion law (Fig. 4),

$$\omega = v_F q_x, \tag{5.3}$$

$$\theta_N(q_x,\omega) = -\theta_{\bar{N}}(q_x,\omega). \tag{5.4}$$

As expected from the mean-field analysis (Sec. III), the Goldstone mode corresponds to an out-of-phase oscillation of the two SDW's. We do not find any renormalization of the mode velocity because we have not taken into account long-wavelength charge fluctuations. The latter couple to the sliding modes and renormalize the velocity to $v_F[1+g_4N(0)]^{1/2}$, where g_4 is the forward scattering strength.[1] In order to obtain this velocity renormalization, it would have been necessary to include forward scattering in the interaction Hamiltonian and to introduce an auxiliary field for the long-wavelength charge fluctuations.[2]

B. Gapped mode

The second mode obtained from Eq. (5.2) corresponds to a gapped mode with dispersion law (Fig. 4)

$$\omega^2 = v_F^2 q_x^2 + \omega_0^2, \tag{5.5}$$

$$\omega_0^2 = \frac{12}{\tilde{g}_2}\,\frac{r}{1-r^2}\,\frac{3+5\tilde{\gamma}^2}{3\tilde{\gamma}^2}\left|\frac{\tilde{\Delta}_N\tilde{\Delta}_{\bar{N}}}{I_N I_{\bar{N}}}\right|. \tag{5.6}$$

The oscillations of this mode satisfy

$$\frac{\theta_N}{\theta_{\bar{N}}} = \frac{r\zeta-\tilde{\gamma}}{r\tilde{\gamma}-\zeta}\,\frac{r(3+\tilde{\gamma}^2+4\tilde{\gamma}\zeta)^2 + (1-r)^2 4\zeta\tilde{\gamma}(3+\tilde{\gamma}^2)}{(3+\tilde{\gamma}^2+4r\tilde{\gamma}\zeta)^2}. \tag{5.7}$$

If we consider the case where the two SDW's have the same amplitude (which is not physical because of the appearance of the helicoidal phase), which occurs when $\tilde{\gamma}=\zeta=\pm1$, we obtain the simple result $\theta_N = \theta_{\bar{N}}$.

Using the physical parameters of the Bechgaard salts, we find that ω_0 is generally larger than the order parameters $|\tilde{\Delta}_{\pm N}|$, so that the gapped sliding mode appears above the quasiparticle excitation gap (in general, within the first Landau subband above the Fermi level). We therefore expect this mode to be strongly damped due to the coupling with the quasiparticle excitations. Note that Eq. (5.6) giving ω_0 is correct only if $\omega_0 \ll |\tilde{\Delta}_N|(1-\tilde{\gamma}^2)^{1/2}$, since it was obtained using expressions of the susceptibilities $c_{pp'}$ valid in the limit $\omega_\nu^2, v_F^2 q_x^2 \ll \tilde{\Delta}_N^2(1-\tilde{\gamma}^2)$.

It is worth pointing out that the two sliding modes bear some similarities with phase modes occurring in two-band[11] or bilayer[12] superconductors. [This also holds for the spin-wave modes (Sec. VII) and the collective modes of the helicoidal phase (Sec. IX).] g_3/g_2 plays the same role as the ratio between the interband (or interlayer) and intraband (or intralayer) coupling constants. There are also analogies with phase modes in $d+id'$ superconductors, and, to some extent, with plasmon modes occurring in conducting bilayer systems.[14] While the corresponding phase modes in a superconducting systems have not yet been observed, plasmon modes in a semiconductor double-well structure have been observed recently via inelastic-light-scattering experiments.[37]

VI. OPTICAL CONDUCTIVITY

At $T=0$ there are no thermally excited carriers above the SDW gap. The response to a low-frequency electric field is then entirely due to fluctuations of the SDW condensate. Since the electric field does not couple to the amplitude modes, we only have to consider the phase fluctuations studied in the preceding sections. In the next section, we determine the charge ($\rho_{\rm DW}$) and current ($j_{\rm DW}$) fluctuations induced by phase fluctuations of the condensate. In Sec. VI B, we calculate the current-current correlation function $\langle j_{\rm DW} j_{\rm DW}\rangle$ which determines the conductivity.

A. Charge and current operators

In this section we calculate explicitly the charge induced by phase fluctuations of the condensate. The current is then obtained from the continuity equation.

We consider a source field $A_0(\mathbf{r},\tau)$ which couples to the (long-wavelength) charge density fluctuations. This adds to the fermionic action the contribution

$$\Delta S[\psi^*,\psi]=\int d^2r\,d\tau A_0(\mathbf{r},\tau)\sum_{\alpha,\sigma}\psi^*_{\alpha\sigma}(\mathbf{r},\tau)\psi_{\alpha\sigma}(\mathbf{r},\tau). \tag{6.1}$$

The charge operator is then obtained from the action by functional differentiation with respect to the external field A_0:

$$\rho(\mathbf{r},\tau)=\frac{\delta S}{\delta A_0(\mathbf{r},\tau)}\Big|_{A_0=0}. \tag{6.2}$$

Following Sec. II, we introduce the auxiliary field $\Delta_{\alpha\sigma}(\mathbf{r},\tau)$ and integrate out the fermions. This leads to the action

$$S[\Delta^*,\Delta,A_0]=\int d^2r\,d\tau\Delta^\dagger_\uparrow(\mathbf{r},\tau)\hat g(\mathbf{r})\Delta_\uparrow(\mathbf{r},\tau) -\sum_\alpha \mathrm{Tr}\ln(-\mathcal{G}^{-1}_\alpha+\hat A_0), \tag{6.3}$$

where

$$\mathcal{G}^{-1}_\alpha(\mathbf{r},\tau;\mathbf{r}',\tau')=\begin{pmatrix} G^{(0)-1}_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau') & \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau')\tilde\Delta_{\alpha\uparrow}(\mathbf{r},\tau)\\ \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau')\tilde\Delta^*_{\alpha\uparrow}(\mathbf{r},\tau) & G^{(0)-1}_{\alpha\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau')\end{pmatrix}, \tag{6.4}$$

$$\hat A_0(\mathbf{r},\tau;\mathbf{r}',\tau')=\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau')A_0(\mathbf{r},\tau)\hat 1. \tag{6.5}$$

We denote by $\hat 1$ the 2×2 unit matrix. Expanding the action with respect to A_0, we obtain

$$S[\Delta^*,\Delta,A_0]=S[\Delta^*,\Delta]+\sum_\alpha \mathrm{Tr}[\mathcal{G}_\alpha\hat A_0]+O(A_0^2), \tag{6.6}$$

where $S[\Delta^*,\Delta]$ is the action without the source field. Here Tr denotes the trace with respect to time, space, and matrix indices. From Eq. (6.2), we then obtain the following expression of the charge operator:

$$\rho(\mathbf{r},\tau)=\sum_\alpha \mathrm{tr}\,\mathcal{G}_\alpha(\mathbf{r},\tau;\mathbf{r},\tau), \tag{6.7}$$

where tr denotes the trace with respect to the matrix indices only [i.e., $\mathrm{Tr}\,\hat O=\int d^2r\,d\tau\,\mathrm{tr}\,\hat O(\mathbf{r},\tau;\mathbf{r},\tau)$ for any operator $\hat O$].

To calculate ρ, we write

$$\tilde\Delta_\sigma(\mathbf{r},\tau)=\tilde\Delta_\sigma(\mathbf{r})+\tilde\eta_\sigma(\mathbf{r},\tau),$$

$$\tilde\eta_\sigma(\mathbf{r},\tau)=\hat g(\mathbf{r})\,\eta_\sigma(\mathbf{r},\tau). \tag{6.8}$$

Then we have

$$\mathcal{G}^{-1}_\alpha=\mathcal{G}^{\mathrm{MF}-1}_\alpha-\Sigma_\alpha, \tag{6.9}$$

$$\Sigma_\alpha(\mathbf{r},\tau;\mathbf{r}',\tau')=-\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau') \times\begin{pmatrix}0 & \tilde\eta_{\alpha\uparrow}(\mathbf{r},\tau)\\ \tilde\eta^*_{\alpha\uparrow}(\mathbf{r},\tau) & 0\end{pmatrix}, \tag{6.10}$$

where $\mathcal{G}^{\mathrm{MF}}$ is the mean-field propagator. The charge operator is given by

$$\rho(\mathbf{r},\tau)=\sum_\alpha \mathrm{tr}\,\mathcal{G}^{\mathrm{MF}}_\alpha(\mathbf{r},\tau;\mathbf{r},\tau) +\sum_\alpha \mathrm{tr}[\mathcal{G}^{\mathrm{MF}}_\alpha\Sigma_\alpha\mathcal{G}^{\mathrm{MF}}_\alpha](\mathbf{r},\tau;\mathbf{r},\tau)+O(\tilde\eta^2). \tag{6.11}$$

The first term in the right-hand side of Eq. (6.11) is the uniform charge density ρ_0 in the mean-field state. The other terms correspond to charge fluctuations $\rho_{\rm DW}$ induced by the condensate fluctuations. To lowest order in phase fluctuations, we obtain

$$\rho_{\rm DW}(\mathbf{r},\tau)=\sum_\alpha \mathrm{tr}[\mathcal{G}^{\mathrm{MF}}_\alpha\Sigma_\alpha\mathcal{G}^{\mathrm{MF}}_\alpha](\mathbf{r},\tau;\mathbf{r},\tau). \tag{6.12}$$

Considering only phase fluctuations [Eq. (4.4)], we obtain (Appendix B)

$$\rho_{\rm DW}(\mathbf{r},\tau)=\frac{1}{\pi b}\partial_x\tilde\theta_N(\mathbf{r},\tau), \tag{6.13}$$

where

$$\tilde{\theta}_N(\mathbf{r},\tau)=\tfrac{1}{2}[\tilde{\varphi}^{(N)}_{+\uparrow}(\mathbf{r},\tau)-\tilde{\varphi}^{(N)}_{-\uparrow}(\mathbf{r},\tau)], \tag{6.14}$$

$$\tilde{\varphi}^{(pN)}_{\alpha\sigma}(\mathbf{r},\tau)=\frac{g_2\Delta_{pN}\varphi^{(pN)}_{\alpha\sigma}(\mathbf{r},\tau)+g_3\Delta_{\bar{p}N}\varphi^{(\bar{p}N)}_{\alpha\sigma}(\mathbf{r},\tau)}{g_2\Delta_{pN}+g_3\Delta_{\bar{p}N}}. \tag{6.15}$$

Note that $\rho_{\rm DW}(\mathbf{r},\tau)$ depends only on $\tilde{\theta}_N(\mathbf{r},\tau)$, i.e., on the largest component of the effective potential $\tilde{\Delta}_{\alpha\sigma}(\mathbf{r},\tau)$.

Using the continuity equation (in imaginary time)

$$ie\partial_\tau\rho_{\rm DW}(\mathbf{r},\tau)+\partial_x j_{\rm DW}(\mathbf{r},\tau)=0, \tag{6.16}$$

we obtain the current fluctuations induced by the phase fluctuations:

$$j_{\rm DW}(\mathbf{r},\tau)=-\frac{ie}{\pi b}\partial_\tau\tilde{\theta}_N(\mathbf{r},\tau). \tag{6.17}$$

Note that the current $j_{\rm DW}$ is parallel to the chains. Condensate fluctuations do not induce current fluctuations in the transverse direction.

This simple result [Eqs. (6.13) and (6.17)] can be understood as follows. For a system with a single SDW, the charge fluctuation $\rho_{\rm DW}=\partial_x\theta/\pi b$ is obtained by requiring the SDW gap to remain tied to the Fermi level. Let us briefly recall this argument. A fluctuating SDW potential $\Delta_{\alpha\sigma}(x,\tau)=\Delta_{\alpha\sigma}e^{i\alpha 2k_F x+i\varphi_{\alpha\sigma}(x,\tau)}\simeq\Delta_{\alpha\sigma}e^{i\alpha 2k_F x+ix\partial_x\varphi_{\alpha\sigma}(x,\tau)}$ couples the state $(k_x,\downarrow)$ to $(k_x+2k_F+\partial_x\varphi_{+\uparrow}(x,\tau),\uparrow)$, and $(k_x,\uparrow)$ to $(k_x+2k_F-\partial_x\varphi_{-\uparrow}(x,\tau),\downarrow)$. (For simplicity, we neglect the transverse direction.) The resulting gap will open at the Fermi level only if the system modifies its density in such a way that

$$\delta k_{F+\uparrow}(x,\tau)+\delta k_{F-\downarrow}(x,\tau)=\partial_x\varphi_{+\uparrow}(x,\tau),$$

$$\delta k_{F+\downarrow}(x,\tau)+\delta k_{F-\uparrow}(x,\tau)=-\partial_x\varphi_{-\uparrow}(x,\tau). \tag{6.18}$$

Here we denote by $\alpha[k_F+\delta k_{F\alpha\sigma}(x,\tau)]$ the Fermi wave vector on the $(\alpha\sigma)$ branch of the spectrum. This Fermi wave vector is time and space dependent because of the density fluctuations. Equations (6.18) imply the charge-density variation

$$\begin{aligned}\delta\rho(x,\tau)&=\frac{1}{2\pi b}\sum_\sigma[\delta k_{F+\sigma}(x,\tau)+\delta k_{F-\sigma}(x,\tau)]\\&=\frac{1}{2\pi b}\partial_x[\varphi_{+\uparrow}(x,\tau)-\varphi_{-\uparrow}(x,\tau)]\\&=\frac{1}{\pi b}\partial_x\theta(x,\tau).\end{aligned} \tag{6.19}$$

Let us now go back to the case of interest. We should consider the effective potential $\tilde{\Delta}_{\alpha\sigma}(\mathbf{r},\tau)$, since it is $\tilde{\Delta}_{\alpha\sigma}(\mathbf{r},\tau)$ [and not $\Delta_{\alpha\sigma}(\mathbf{r},\tau)$] that couples to the electrons. In the presence of two SDW's, it is not possible to satisfy Eq. (6.18) for the two components (N and $-N$) of the effective potential. Nevertheless, it is natural to assume that the largest gap remains tied to the Fermi level. This immediately yields Eq. (6.13), and, from the continuity equation, Eq. (6.17). (The latter can also be obtained by a similar argument without using the continuity equation.[1]) In conclusion, Eqs. (6.13) and (6.17) express the fact that the largest gap remains tied to the Fermi level.

Finally, we can express the charge and current fluctuations directly in terms of θ_N and $\theta_{\bar{N}}$:

$$\rho_{\rm DW}(\mathbf{r},\tau)=\frac{1}{\pi b}\frac{\partial_x\theta_N(\mathbf{r},\tau)-r\gamma\partial_x\theta_{\bar{N}}(\mathbf{r},\tau)}{1+r\gamma},$$

$$j_{\rm DW}(\mathbf{r},\tau)=-\frac{ie}{\pi b}\frac{\partial_\tau\theta_N(\mathbf{r},\tau)-r\gamma\partial_\tau\theta_{\bar{N}}(\mathbf{r},\tau)}{1+r\gamma}. \tag{6.20}$$

B. Current-current correlation function

In this section we calculate the correlation function

$$\Pi(\omega_\nu)=\langle j_{\rm DW}(\omega_\nu)j_{\rm DW}(-\omega_\nu)\rangle, \tag{6.21}$$

where $j_{\rm DW}(\omega_\nu)\equiv j_{\rm DW}(\mathbf{q}=0,\omega_\nu)$ is the Fourier transform of the current operator given by Eq. (6.17). We have

$$\Pi(\omega_\nu)=\frac{-e^2\omega_\nu^2}{\pi^2b^2(1+\gamma r)^2}\times[\mathcal{D}^{++}_{\rm ch}(\omega_\nu)+r^2\gamma^2\mathcal{D}^{--}_{\rm ch}(\omega_\nu)-2r\gamma\mathcal{D}^{+-}_{\rm ch}(\omega_\nu)], \tag{6.22}$$

where $\mathcal{D}^{pp'}_{\rm ch}(\omega_\nu)=\langle\theta_{pN}(\omega_\nu)\theta_{p'N}(-\omega_\nu)\rangle$ [see Eq. (4.8)].

The conductivity is defined by $\sigma(\omega)=-i/(\omega+i0^+)\Pi^{\rm ret}(\omega)$ where the retarded correlation function $\Pi^{\rm ret}(\omega)$ is obtained from $\Pi(\omega_\nu)$ by analytical continuation to real frequency $i\omega_\nu\to\omega+i0^+$. Using the expression of $c_{pp'}$ (Appendix A), we obtain the dissipative part of the conductivity,

$$\mathrm{Re}[\sigma(\omega)]=\frac{\omega_p^2}{4}\left(\delta(\omega)\frac{3(1-\tilde{\gamma}^2)}{3+5\tilde{\gamma}^2}+\delta(\omega\pm\omega_0)\frac{4\tilde{\gamma}}{3+5\tilde{\gamma}^2}\right), \tag{6.23}$$

where $\omega_p=\sqrt{8e^2v_F/b}$ is the plasma frequency. Equation (6.23) satisfies the conductivity sum rule

$$\int_{-\infty}^{\infty}d\omega\,\mathrm{Re}[\sigma(\omega)]=\frac{\omega_p^2}{4}. \tag{6.24}$$

Thus all the spectral weight is exhausted by the collective sliding modes. Quasiparticle excitations above the mean-field gap do not contribute to the (longitudinal) optical conductivity, a result well known in SDW systems.[1] Because both modes contribute to the conductivity, the low-energy (Goldstone) mode carries only a fraction of the total spectral weight. We obtain Dirac peaks at $\pm\omega_0$ because we have neglected the coupling of the gapped mode with quasiparticle excitations. Also, in a real system (with impurities), the Goldstone mode would broaden and appear at a finite frequency (below the quasiparticle excitation gap) due to pinning by impurities. In the clean limit, which is appropriate in $(\mathrm{TMTSF})_2X$ salts, the presence of impurities does not restore any significant spectral weight to quasiparticle excitations above the mean-field gap.[1] Therefore, the fraction of spectral weight carried by the two modes is correctly given

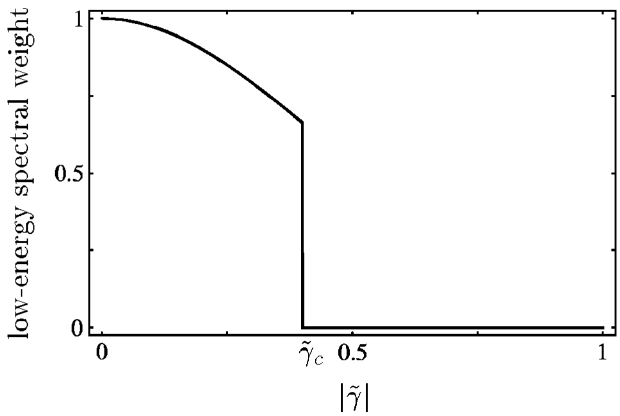

FIG. 5. Fraction of the total optical spectral weight $\omega_p^2/4$ carried by the low-energy collective modes. In the sinusoidal phase ($|\tilde{\gamma}| \leqslant \tilde{\gamma}_c \sim 0.4$), the low-energy sliding mode carries the fraction $3(1-\tilde{\gamma}^2)/(3+5\tilde{\gamma}^2)$ of the total spectral weight. In the helicoidal phase, there is no spectral weight at low energy (see Sec. IX).

by Eq. (6.23). By measuring the optical conductivity $\sigma(\omega)$, we can therefore obtain the ratio $|\tilde{\gamma}| \simeq |\gamma|$ of the amplitudes of the two SDW's.

We have shown in Ref. 21 that the sinusoidal phase becomes unstable against the formation of a helicoidal phase when $|\tilde{\gamma}|$ reaches a critical value $\tilde{\gamma}_c$. By comparing the mean-field condensation energies of the sinusoidal [Eq. (3.12)] and helicoidal [Eq. (9.5) in Sec. IX below] phases, we find $\tilde{\gamma}_c \sim 0.4$, which is in good agreement with the result $\tilde{\gamma}_c \sim 0.52$ obtained from the Ginzburg-Landau expansion of the free energy close to the transition line.[21]

Since $|\tilde{\gamma}|$ varies between 0 and 0.4 in the sinusoidal phase, the Goldstone mode can carry between $\sim 100\%$ and $\sim 50\%$ of the total spectral weight (Fig. 5). When this fraction is reduced below $\sim 50\%$ (i.e., when $|\tilde{\gamma}| \gtrsim 0.4$), a first order transition to the helicoidal phase takes place. In the helicoidal phase, there is only one Goldstone mode (see Sec. IX). This mode is a pure spin-wave mode and does not contribute to the optical spectral weight.

Experimentally, the helicoidal phase can be stabilized by decreasing pressure, which increases the ratio $r = g_3/g_2$ (Fig. 2).[21] Figure 6 shows $|\tilde{\gamma}|$ and the low-energy spectral weight as a function of r for $H \simeq 10$ T. In the helicoidal phase, $\tilde{\gamma}$ is defined by Eq. (9.8).

VII. SPIN-WAVE MODES

For the spin-wave modes, the condition $\det \mathcal{D}_{\rm sp}^{-1}(\tilde{q}) = 0$ yields

$$\delta c_{++} \frac{r\zeta}{\tilde{\gamma}} + \delta c_{--} \frac{r\tilde{\gamma}}{\zeta} + 2r\,\delta c_{+-} - (1-r^2)(\delta c_{++}\delta c_{--} - \delta c_{+-}^2) = 0. \tag{7.1}$$

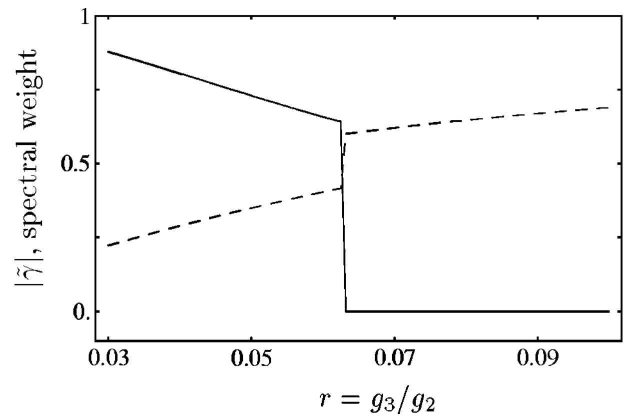

FIG. 6. $|\tilde{\gamma}|$ (dashed line) and low-energy optical spectral weight (solid line) as a function of $r = g_3/g_2$ in the phase $N=-2$ ($H = 10$ T). The transition from the sinusoidal to the helicoidal phase occurs for $r \sim 0.06$.

A. Goldstone mode

From Eq. (7.1) we deduce the existence of a mode with a linear dispersion law

$$\omega = v_F q_x, \tag{7.2}$$

$$\phi_N(q_x,\omega) = \phi_{\bar{N}}(q_x,\omega). \tag{7.3}$$

The spin-wave Goldstone mode corresponds to in-phase oscillations of the two SDW's, in agreement with the conclusion of the mean-field analysis (Sec. III). We do not find any renormalization of its velocity because we have not taken into account the coupling with the long-wavelength spin fluctuations.[1]

B. Gapped mode

The other solution of Eq. (7.1) corresponds to a gapped mode with the dispersion law

$$\omega^2 = v_F^2 q_x^2 + \omega_1^2,$$

$$\omega_1^2 = \frac{12}{\tilde{g}_2} \frac{r}{1-r^2} \frac{1-\tilde{\gamma}^2}{\tilde{\gamma}^2} \left| \frac{\tilde{\Delta}_N \tilde{\Delta}_{\bar{N}}}{I_N I_{\bar{N}}} \right|. \tag{7.4}$$

The oscillations of this mode satisfy

$$\phi_N(q_x,\omega) = -r\gamma\phi_{\bar{N}}(q_x,\omega). \tag{7.5}$$

In the gapped mode, the oscillations of the two SDW's are out of phase.[38] As expected, the larger SDW has the smaller oscillations. Like the gapped sliding mode, this mode is found to lie in general above the quasiparticle excitation gap, and is therefore expected to be strongly damped due to coupling with quasiparticle excitations. Thus, the spin-wave modes are similar to the sliding modes as shown in Fig. 4.

VIII. SPIN-SPIN CORRELATION FUNCTION

In this section we calculate the transverse spin-spin correlation function. For real order parameters $\Delta_{\alpha\sigma}^{(pN)}$, the mean-field magnetization $\langle \mathbf{S}(\mathbf{r}) \rangle$ is parallel to the x axis. Transverse (to the magnetization) spin fluctuations correspond to fluctuations of the operator

$$S_y(\mathbf{r},\tau)=\frac{i}{2}\sum_{\alpha,\sigma}\sigma O_{\alpha\sigma}(\mathbf{r},\tau). \tag{8.1}$$

The correlation function $\chi_{yy}=\langle S_y S_y\rangle$ can be expressed as a functional derivative of the free energy with respect to an external field that couples to the spin operator S_y. This standard procedure allows us to rewrite χ_{yy} as

$$\chi_{yy}(\mathbf{r},\tau;\mathbf{r}',\tau')=\frac{1}{4}\sum_{\alpha,\alpha',\sigma,\sigma'}\sigma\sigma'\langle\Delta_{\alpha\sigma}(\mathbf{r},\tau)\Delta^*_{\alpha'\sigma'}(\mathbf{r}',\tau')\rangle -\frac{1}{2}\delta(\mathbf{r}-\mathbf{r}')\,\delta(\tau-\tau')\sum_{\alpha,\alpha'}\hat{g}^{-1}_{\alpha\alpha'}(\mathbf{r}), \tag{8.2}$$

where the mean value $\langle\cdots\rangle$ is taken with the effective action $S[\Delta^*,\Delta]$. In the following, we drop the last term, which does not contribute to the spectral function $\mathrm{Im}\,\chi^{\mathrm{ret}}_{yy}$. To first order in phase fluctuations,

$$\chi_{yy}(\mathbf{r},\tau;\mathbf{r}',\tau')=4\sum_{p,p'}\Delta_{pN}\Delta_{p'N}\cos(\mathbf{Q}_{pN}\cdot\mathbf{r})\times\cos(\mathbf{Q}_{p'N}\cdot\mathbf{r}')\mathcal{D}^{pp'}_{\mathrm{sp}}(\mathbf{r},\tau;\mathbf{r}',\tau'), \tag{8.3}$$

where $\mathcal{D}^{pp'}_{\mathrm{sp}}(\mathbf{r},\tau;\mathbf{r}',\tau')=\langle\phi_{pN}(\mathbf{r},\tau)\phi_{p'N}(\mathbf{r}',\tau')\rangle$. Taking the Fourier transform, we obtain

$$\chi_{yy}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}+\alpha'\mathbf{Q}_{p'N},\omega_\nu)=\Delta_{pN}\Delta_{p'N}\mathcal{D}^{pp'}_{\mathrm{sp}}(\tilde{q}), \tag{8.4}$$

where $\mathbf{q}=(q_x,q_y=0)$. Because of the presence of SDW's, the spin-spin correlation function is not diagonal in momentum space. In Eq. (8.4), $\tilde{q}$ corresponds to the momentum and energy of the spin-wave mode and tends to zero for long-wavelength fluctuations. We therefore consider the spectral function $\mathrm{Im}\,\mathrm{Tr}_{\mathbf{q}}\chi^{\mathrm{ret}}_{yy}$, where $\mathrm{Tr}_{\mathbf{q}}$ is a partial trace corresponding to a given spin-wave mode momentum $\mathbf{q}=(q_x,q_y=0)$:

$$\mathrm{Tr}_{\mathbf{q}}\chi_{yy}=\sum_{\alpha,p}\chi_{yy}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}+\alpha\mathbf{Q}_{pN},\omega_\nu) =2\Delta_N^2[\mathcal{D}^{++}_{\mathrm{sp}}(\tilde{q})+\gamma^2\mathcal{D}^{--}_{\mathrm{sp}}(\tilde{q})]. \tag{8.5}$$

Using the expression of $c_{pp'}$ (Appendix A), we obtain

$$\mathrm{Im}\,\mathrm{Tr}_{\mathbf{q}}\chi^{\mathrm{ret}}_{yy}=\frac{2\pi}{g_2^2N(0)}\left|\frac{\tilde{\Delta}_N\tilde{\Delta}_{\bar{N}}}{I_NI_{-N}}\right|\times\left[\frac{\delta(\omega-v_Fq_x)}{v_Fq_x}\left(\frac{-4r}{(1-r^2)^2}+\frac{1+r^2}{(1-r^2)^2}\frac{\tilde{\gamma}^2+\zeta^2}{|\tilde{\gamma}\zeta|}\right)+\frac{\delta(\omega-\omega_1)}{\omega_1}\frac{3(1+r^2)}{2(1-r^2)^2}\frac{1-\tilde{\gamma}^2}{|\tilde{\gamma}\zeta|}\right] \tag{8.6}$$

for $\omega,q_x>0$ and $q_y=0$. Both spin-wave modes contribute to the spectral function. The spectral weight carried by the Goldstone mode diverges as $1/q_x$, as expected for a quantum antiferromagnet.[39]

Equations (6.23) and (8.6) predict that all the spectral weight is carried by the in-phase modes, i.e., the gapped sliding mode and the gapless spin-wave mode, whenever both SDW's have the same amplitude ($\tilde{\gamma}=\zeta=\pm1$).

IX. HELICOIDAL PHASE

The analysis of the helicoidal phase turns out to be much simpler than that of the sinusoidal phase. In the helicoidal phase, the mean-field gap does not depend on the transverse momentum k_y, which significantly simplifies the computation of the collective modes. In this section, we shall describe the properties of the helicoidal phase, but skipping most of the technical details of the derivation.

A. Mean-field theory

The helicoidal phase is characterized by the order parameter[21]

$$\Delta_{\alpha\sigma}(\mathbf{r})=\langle O_{\alpha\sigma}(\mathbf{r},\tau)\rangle=\Delta_{\alpha\sigma}e^{i\alpha\mathbf{Q}_{p(\alpha,\sigma)N}\cdot\mathbf{r}}, \tag{9.1}$$

with

$$p(+,\uparrow)=p(-,\downarrow)=+,\quad p(+,\downarrow)=p(-,\uparrow)=-. \tag{9.2}$$

In the notations of the preceding sections, this corresponds to $\Delta^{(N)}_{+\uparrow}\equiv\Delta_{+\uparrow}=\Delta^*_{-\downarrow}$, $\Delta^{(\bar{N})}_{+\downarrow}\equiv\Delta_{+\downarrow}=\Delta^*_{-\uparrow}$, $\Delta^{(N)}_{+\downarrow}=\Delta^{(N)}_{-\uparrow}=0$, and $\Delta^{(\bar{N})}_{+\uparrow}=\Delta^{(\bar{N})}_{-\downarrow}=0$. The fact that some order parameters vanish in the helicoidal phase makes the computation of the collective modes much simpler. The spin-density modulation is given by[21]

$$\begin{aligned}\langle S_x(\mathbf{r})\rangle&=2|\Delta_{+\uparrow}|\cos(-\mathbf{Q}_N\cdot\mathbf{r}-\varphi_{+\uparrow})+2|\Delta_{-\uparrow}|\cos(\mathbf{Q}_{\bar{N}}\cdot\mathbf{r}-\varphi_{-\uparrow}),\\ \langle S_y(\mathbf{r})\rangle&=2|\Delta_{+\uparrow}|\sin(-\mathbf{Q}_N\cdot\mathbf{r}-\varphi_{+\uparrow})+2|\Delta_{-\uparrow}|\sin(\mathbf{Q}_{\bar{N}}\cdot\mathbf{r}-\varphi_{-\uparrow}),\\ \langle S_z(\mathbf{r})\rangle&=0,\end{aligned} \tag{9.3}$$

which corresponds to two helicoidal SDW's with opposite chiralities. The mean-field action is still given by Eq. (3.6), but the pairing amplitudes are given by (in the QLA)

$$\begin{aligned}\tilde{\Delta}_{\alpha\sigma}(\mathbf{k},\mathbf{k}')&=\delta_{\mathbf{k}',\mathbf{k}-\alpha\mathbf{Q}_0}\tilde{\Delta}_{\alpha\sigma}(k_y),\\ \tilde{\Delta}_{\alpha\sigma}(k_y)&=\tilde{\Delta}_{\alpha\sigma}e^{i\alpha p(\alpha,\sigma)N(k_yb-\pi/2)},\\ \tilde{\Delta}_{\alpha\sigma}&=I_{p(\alpha,\sigma)N}(g_2\Delta_{\alpha\sigma}+g_3\Delta_{\bar{\alpha}\sigma}).\end{aligned} \tag{9.4}$$

The ground-state energy reads

$$\Delta E=\sum_\alpha\frac{1}{I_{p(\alpha,\uparrow)N}}\Delta^*_{\alpha\uparrow}\tilde{\Delta}_{\alpha\uparrow}-\frac{N(0)}{2}\sum_\alpha\left(\frac{|\tilde{\Delta}_{\alpha\uparrow}|^2}{2}+|\tilde{\Delta}_{\alpha\uparrow}|^2\ln\frac{2E_0}{|\tilde{\Delta}_{\alpha\uparrow}|}\right)=-\frac{1}{4}N(0)\sum_\alpha|\tilde{\Delta}_{\alpha\uparrow}|^2, \tag{9.5}$$

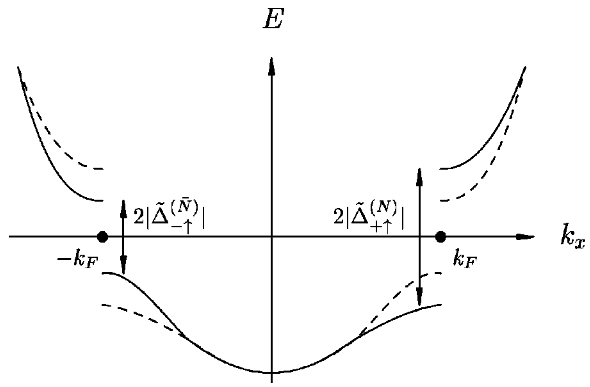

FIG. 7. Quasiparticle excitation spectrum in the helicoidal phase. The solid (dashed) line corresponds to up (down) spins. For clarity, we have not shown the Zeeman splitting.

where the last line is obtained using the gap equation $\partial \Delta E/\partial \tilde{\Delta}^*_{\alpha\uparrow}=0$. From the latter, we also deduce (writing $\tilde{\Delta}_{\alpha\sigma}=|\tilde{\Delta}_{\alpha\sigma}|e^{i\tilde{\varphi}_{\alpha\sigma}}$)

$$\tilde{\varphi}_{+\uparrow}-\tilde{\varphi}_{-\uparrow}=\begin{cases}0 & \text{if } I_N I_{\bar{N}}>0\\ \pi & \text{if } I_N I_{\bar{N}}<0,\end{cases} \tag{9.6}$$

and

$$\varphi_{+\uparrow}=\varphi_{-\uparrow}. \tag{9.7}$$

Without loss of generality, we can choose the order parameters real, and introduce

$$\gamma=\frac{\Delta_{-\uparrow}}{\Delta_{+\uparrow}}, \quad \tilde{\gamma}=\frac{\tilde{\Delta}_{-\uparrow}}{\tilde{\Delta}_{+\uparrow}}. \tag{9.8}$$

γ and $\tilde{\gamma}$ satisfy Eqs. (3.20).

The mean-field propagators are then

$$G_{\alpha\sigma}(\mathbf{k},\omega)=\frac{-i\omega-\epsilon_{\alpha\sigma}(k_x)}{\omega^2+\epsilon^2_{\alpha\sigma}(k_x)+|\tilde{\Delta}_{\alpha\sigma}|^2},$$

$$F_{\alpha\sigma}(\mathbf{k},\omega)=\frac{\tilde{\Delta}_{\alpha\sigma}e^{i\alpha p(\alpha,\sigma)N(k_y b-\pi/2)}}{\omega^2+\epsilon^2_{\alpha\sigma}(k_x)+|\tilde{\Delta}_{\alpha\sigma}|^2}. \tag{9.9}$$

The excitation energy $E_{\alpha\sigma}(\mathbf{k})=[\epsilon^2_{\alpha\sigma}(k_x)+|\tilde{\Delta}_{\alpha\sigma}|^2]^{1/2}$ does not depend on k_y. The spectrum is shown in Fig. 7. Note that all the branches are gapped, due to the presence of two SDW's. In contrast, in the presence of a single helicoidal SDW, some branches would remain gapless.

As in the sinusoidal phase, we can compute the mean-field susceptibilities. They are given in Appendix C.

B. Collective modes

Considering phase fluctuations only, we have

$$\eta_{\alpha\sigma}(\mathbf{r},\tau)=\Delta_{\alpha\sigma}e^{i\alpha\mathbf{Q}_{p(\alpha,\sigma)N}\cdot\mathbf{r}+i\varphi_{\alpha\sigma}(\mathbf{r},\tau)}-\Delta_{\alpha\sigma}(\mathbf{r},\tau)$$
$$\simeq i\Delta_{\alpha\sigma}(\mathbf{r})\varphi_{\alpha\sigma}(\mathbf{r},\tau) \tag{9.10}$$

to lowest order in phase fluctuations. Following Sec. IV, we then obtain the effective action

$$S[\varphi]=\frac{1}{2}\sum_{\tilde{q}}(\varphi_{+\uparrow}(-\tilde{q}),\varphi_{-\uparrow}(-\tilde{q}))\mathcal{D}^{-1}(\tilde{q})\begin{pmatrix}\varphi_{+\uparrow}(\tilde{q})\\ \varphi_{-\uparrow}(\tilde{q})\end{pmatrix}, \tag{9.11}$$

$$\mathcal{D}^{-1}(\tilde{q})=2g_2\Delta^2_{+\uparrow}\times\begin{pmatrix}1-c_{++}-r^2c_{--} & \gamma r(1-c_{++}-c_{--})\\ \gamma r(1-c_{++}-c_{--}) & \gamma^2(1-c_{--}-r^2c_{++})\end{pmatrix}. \tag{9.12}$$

The dispersion of the collective modes is obtained from $\det \mathcal{D}^{-1}(\tilde{q})=0$. We find a Goldstone mode satisfying

$$\omega=v_F q_x, \quad \varphi_{+\uparrow}=\varphi_{-\uparrow}. \tag{9.13}$$

This mode corresponds to a uniform spin rotation. Equivalently, it can also be seen as a translation of the two SDW's in opposite directions.[40] Thus it combines characteristics of the two Goldstone modes of the sinusoidal phase.

There is also a gapped mode with dispersion law $\omega^2=v_F^2q_x^2+\omega_2^2$, where

$$\omega_2^2=\frac{16}{\tilde{g}_2}\frac{r}{1-r^2}\left|\frac{\tilde{\Delta}_{+\uparrow}\tilde{\Delta}_{-\uparrow}}{I_N I_{\bar{N}}}\right|. \tag{9.14}$$

The oscillations satisfy

$$\frac{\varphi_{+\uparrow}}{\varphi_{-\uparrow}}=\frac{\gamma^2(1-r^2)^2+2r\gamma(\gamma+r)^2+2r\gamma(1+\gamma r)^2}{\gamma^2(1-r^2)^2-2(\gamma+r)^2-2r^2(1+\gamma r)^2}. \tag{9.15}$$

When the two SDW's have the same amplitude $(\gamma=1)$, $\varphi_{+\uparrow}/\varphi_{-\uparrow}=-1$. As for the sinusoidal phase, the gapped mode is found to lie above the quasiparticle excitation gap.

The similarities with phase modes in two-band or bilayer superconductors are more pronounced in the helicoidal phase than in the sinusoidal phase. In fact, Eq. (9.14) gives exactly the gap of the phase modes in these superconducting systems. In the helicoidal phase, the order parameter $\Delta_{\alpha\sigma}(\mathbf{r})$ [Eq. (9.1)] and the effective potential $\tilde{\Delta}_{\alpha\sigma}(\mathbf{r})=g_2\Delta_{\alpha\sigma}(\mathbf{r})+g_3e^{i\alpha 4k_F x}\Delta_{\bar{\alpha}\bar{\sigma}}(\mathbf{r})$ have only one component $\sim e^{i\alpha\mathbf{Q}_{p(\alpha,\sigma)N}\cdot\mathbf{r}}$. Thus, the helicoidal phase is more BCS-like than the sinusoidal phase. This explains why the quasiparticle excitation spectrum does not depend on k_y, and also the absence of a $\tilde{\gamma}$-dependent factor in the energy of the gapped mode [Eq. (9.14)]. We have shown in Sec. VI that in the sinusoidal phase only the largest gap remains tied to the Fermi level in the presence of condensate fluctuations. This yields a modification of the condensation energy, which is at the origin of the $\tilde{\gamma}$-dependent factors in the expressions of ω_0 and ω_1.

C. Spectral functions

In the helicoidal phase, it is not clear whether the collective modes should be seen as sliding or spin-wave modes.[40] In this section, we compute the spectral functions $\mathrm{Re}[\sigma(\omega)]$ and $\mathrm{Im}\,\mathrm{Tr}_{\mathbf{q}}\chi^{\mathrm{ret}}$ to answer this question.

1. Optical conductivity

The charge and current operators are given by

$$\rho_{\rm DW}(\mathbf{r},\tau)=\frac{1}{2\pi b}\,\partial_x[\tilde{\varphi}_{+\uparrow}(\mathbf{r},\tau)-\tilde{\varphi}_{-\uparrow}(\mathbf{r},\tau)], \quad (9.16)$$

$$j_{\rm DW}(\mathbf{r},\tau)=-\frac{ie}{2\pi b}\,\partial_\tau[\tilde{\varphi}_{+\uparrow}(\mathbf{r},\tau)-\tilde{\varphi}_{-\uparrow}(\mathbf{r},\tau)], \quad (9.17)$$

where

$$\tilde{\varphi}_{\alpha\sigma}(\mathbf{r},\tau)=\frac{g_2\Delta_{\alpha\sigma}\varphi_{\alpha\sigma}(\mathbf{r},\tau)+g_3\Delta_{\bar{\alpha}\sigma}\varphi_{\bar{\alpha}\sigma}(\mathbf{r},\tau)}{g_2\Delta_{\alpha\sigma}+g_3\Delta_{\bar{\alpha}\sigma}} \quad (9.18)$$

is the fluctuating phase of the effective potential $\tilde{\Delta}_{\alpha\sigma}$. From Eq. (9.17) we can calculate the current-current correlation function, which yields the conductivity

$$\mathrm{Re}[\sigma(\omega)]=\frac{\omega_p^2}{8}\,\delta(\omega\pm\omega_2). \quad (9.19)$$

The Goldstone mode does not contribute to the conductivity and is therefore a pure spin-wave mode (in the limit $\mathbf{q}\to 0$). This result could have been anticipated since the condition $\varphi_{+\uparrow}=\varphi_{-\uparrow}$ implies $\tilde{\varphi}_{+\uparrow}=\tilde{\varphi}_{-\uparrow}$ and the vanishing of the current (9.17). Thus all the spectral weight is pushed up above the quasiparticle excitation gap at frequencies of the order of ω_2. This implies that there is no low-energy collective mode corresponding to a uniform sliding of the condensate. Since nonlinear conduction in SDW systems results from the depinning of such a mode (from the impurity potential) above a threshold electric field, we conclude that there is no nonlinear conductivity in the helicoidal phase.

In Ref. 21, we have shown that the helicoidal phase is characterized by a vanishing QHE ($\sigma_{xy}=0$) and a kinetic magnetoelectric effect. By studying the collective modes, we have found a third possibility for the experimental detection of the helicoidal phase, namely, the absence of nonlinear conduction.

2. Spin-spin correlation function

We consider the spin-spin correlation function

$$\chi_{\mu\nu}(\mathbf{r},\tau;\mathbf{r}',\tau')=\langle S_\mu(\mathbf{r},\tau)S_\nu(\mathbf{r}',\tau')\rangle. \quad (9.20)$$

As in Sec. VIII, we shall compute the partial trace $\mathrm{Tr}_{\mathbf{q}}\chi_{\mu\nu}$, where the momentum $\mathbf{q}$ of the collective mode is held fixed. [Again, we drop the last term of Eq. (8.2).] We find

$$\mathrm{Tr}_{\mathbf{q}}\chi_{\mu\mu}=\frac{1}{2}\sum_\alpha \Delta_{\alpha\uparrow}^2\mathcal{D}_{\alpha\alpha}(\tilde{q}),$$

$$\mathrm{Tr}_{\mathbf{q}}\chi_{xy}=0. \quad (9.21)$$

This yields the spectral function

$$\mathrm{Im\,Tr}_{\mathbf{q}}\chi_{\mu\mu}^{\rm ret}=\frac{\pi}{2g_2^2N(0)}\left|\frac{\tilde{\Delta}_{+\uparrow}\tilde{\Delta}_{-\uparrow}}{I_N I_{\bar{N}}}\right|\Bigg[\frac{\delta(\omega-v_Fq_x)}{v_Fq_x}\times\left(-\frac{4r}{(1-r^2)^2}+\frac{1+r^2}{(1-r^2)^2}\,\frac{\tilde{\gamma}^2+\zeta^2}{\tilde{\gamma}\zeta}\right)+\frac{\delta(\omega-\omega_2)}{\omega_2}\left(\frac{4r}{(1-r^2)^2}+\frac{1+r^2}{(1-r^2)^2}\,\frac{\tilde{\gamma}^2+\zeta^2}{\tilde{\gamma}\zeta}\right)\Bigg] \quad (9.22)$$

for $\omega,q_x>0$ and $q_x\to 0$. Both modes contribute to the spectral function. Although the Goldstone mode is a pure spin-wave mode, the gapped mode has characteristics of both a spin wave and a sliding mode, as can be seen from the spectral function.

X. CONCLUSION

We have studied the long-wavelength collective modes in the FISDW phases of quasi-1D conductors, focusing on phases that exhibit a sign reversal of the QHE (Ribault anomaly). We have recently proposed that two SDW's, with wave vectors $\mathbf{Q}_N=(2k_F+NG,Q_y)$ and $\mathbf{Q}_{\bar{N}}=(2k_F-NG,-Q_y)$, coexist in the Ribault phase, as a result of umklapp scattering. When the latter is strong enough, the two SDW's become circularly polarized (helicoidal SDW's). The presence of two SDW's gives rise to a rich structure of collective modes, which strongly depends on the polarization (linear or circular) of the SDW's.

Regarding the sliding modes, we find that the out-of-phase oscillations are gapless in the long-wavelength limit. The fact that this Goldstone mode corresponds to out-of-phase (and not in-phase) oscillations is related to the pinning by the lattice (due to umklapp processes) that would occur for a single commensurate SDW. The other sliding mode is gapped and corresponds to in-phase oscillations. In Bechgaard salts, this mode is expected to lie above the quasiparticle excitation gap and should therefore be strongly damped due to the coupling with the quasiparticle excitations. In the helicoidal phase, there is no low-energy sliding mode, since the Goldstone mode is a pure spin-wave mode. (For a helicoidal SDW, one cannot distinguish between a uniform spin rotation and a global translation, so that we cannot classify the modes into sliding and spin-wave modes.)

The dissipative part of the conductivity, $\mathrm{Re}[\sigma(\omega)]$, exhibits two peaks: a low-energy peak corresponding to the Goldstone mode, and a (broader) peak at high energy due to the incoherent gapped mode. The low-energy spectral weight is directly related to the ratio of the amplitudes of the two SDW's that coexist in the Ribault phase. When the umklapp scattering strength (g_3) increases (experimentally this corresponds to a pressure decrease), spectral weight is transferred from the low-energy peak to high energies. Above a critical value of g_3, the sinusoidal phase becomes unstable with respect to the formation of a helicoidal phase. At the transition, the low-energy spectral weight suddenly drops to zero (Fig. 5), since there is no low-energy optical spectral weight in the helicoidal phase. Thus, the formation of the helicoidal phase can be detected by measuring the low-energy optical spectral weight. We also note that the absence of a low-energy sliding mode means that there is no infinite Fröhlich conductivity[41] in this helicoidal phase, which is therefore a true insulating phase even in an ideal system (i.e., one with no impurities). In a real system (with impurities), this implies the absence of nonlinear dc conductivity.

The spin-wave modes exhibit a similar structure. The in-phase oscillations of the two SDW's are gapless (Goldstone mode), while the out-of-phase oscillations are gapped. In the helicoidal phase, the Goldstone mode is a pure spin-wave mode, but the gapped mode contributes both to the conductivity and to the transverse spin-spin correlation function.

As discussed in the Introduction, these conclusions rely on a simple Fermi surface [Fig. 1(a)], which does not necessarily provide a good approximation to the actual Fermi surface of the Bechgaard salts. They should therefore be taken with caution regarding their relevance to the organic conductors of the Bechgaard salts family. However, we have derived a number of experimental consequences that should allows tests of our theory. In the sinusoidal phase, we predict a possible reentrance of the phase $N=0$ within the cascade.[21] The low-energy peak in the optical conductivity $\mathrm{Re}[\sigma(\omega)]$ carries only a fraction of the total spectral weight $\omega_p^2/4$. This fraction should decrease with pressure. At low pressure, the sinusoidal phase may become helicoidal. The helicoidal phase is characterized by a vanishing QHE,[21] a kinetic magnetoelectric effect,[21] and the absence of low-energy spectral weight in the optical conductivity as well as the absence of nonlinear dc conductivity. In the alternative scenario proposed by Zanchi and Montambaux,[23] the Ribault phase does not exhibit any special features compared to the positive phases, apart from the unusual behavior of the magnetoroton modes,[26] but these modes have not been observed yet.

We expect our conclusions regarding the structure of the collective modes and the associated spectral functions to hold for generic two-SDW systems and not only for the FISDW phases that exhibit the Ribault anomaly. It is clear that the existence of four long-wavelength collective modes (two spin-wave and two sliding modes) results from the presence of a second SDW, which doubles the number of degrees of freedom. Two of these modes should be gapless as expected from the Goldstone theorem in a system where two continuous symmetries are spontaneously broken: the translation symmetry in real space and the rotation symmetry in spin space.

This belief is supported by the striking analogy with collective modes in other systems like two-band, bilayer, and $d+id'$ superconductors,[11–13] and, to a lesser extent, plasmon modes in semiconductor double-well structures.[14] While phase modes are in general difficult to observe in superconductors, since they do not directly couple to external probes (see, however, Ref. 13), collective modes in SDW systems directly show up in response functions like, for instance, the dc and optical conductivities.

Finally, we point out that we expect a similar structure of collective modes in the FISDW phases of the compound $(\mathrm{TMTSF})_2\mathrm{ClO}_4$. Due to anion ordering, the unit cell contains two sites, which leads to two electronic bands at the Fermi level. This doubles the number of degrees of freedom and therefore the number of collective modes (without considering the possible formation of a second SDW due to umklapp scattering). As for phonon modes in a crystal with two molecules per unit cell, we expect an acoustic (Goldstone) mode and an optical mode.

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APPENDIX A: MEAN-FIELD SUSCEPTIBILITIES $\bar{\chi}$ AND $\tilde{\chi}$

Using the expression (3.21) of the mean-field propagators G and F, we obtain

$$\begin{aligned}\bar{\chi}_{\alpha\sigma,\alpha\sigma}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}'+\alpha\mathbf{Q}_{p'N},\omega_\nu)&=-\delta_{\mathbf{q},\mathbf{q}'}\frac{T}{L_xL_y}\sum_{\mathbf{k},\omega}\sum_{n=-\infty}^{\infty}G_{\alpha\sigma}(\mathbf{k},\omega)\\&\times G_{\overline{\alpha\sigma}}(\mathbf{k}-\alpha\mathbf{Q}_0-\mathbf{q}+\alpha nG,\omega-\omega_\nu)\\&\times I_{pN+n}(\pi/b+q_y)I_{p'N+n}(\pi/b+q_y)\\&\times e^{-i\alpha(p-p')N(k_yb-q_yb/2-\pi/2)},\end{aligned}\tag{A1}$$

$$\begin{aligned}\bar{\chi}_{\alpha\sigma,\overline{\alpha\sigma}}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}'-\alpha\mathbf{Q}_{p'N},\omega_\nu)&=-\delta_{\mathbf{q},\mathbf{q}'}\frac{T}{L_xL_y}\sum_{\mathbf{k},\omega}\sum_{n=-\infty}^{\infty}F_{\alpha\sigma}(\mathbf{k},\omega)\\&\times F_{\alpha\sigma}(\mathbf{k}-\mathbf{q}+\alpha nG,\omega-\omega_\nu)I_{pN+n}(\pi/b+q_y)\\&\times I_{p'N-n}(\pi/b+q_y)e^{-i\alpha(p+p')N(k_yb-q_yb/2-\pi/2)}.\end{aligned}\tag{A2}$$

Here we have assumed that $|q_x|,|q_x'|\ll G$, which is the case of interest when studying the low-energy fluctuations around the mean-field solution.

In the QLA, we retain only the term $n=0$ in the above equations, since the other terms are strongly suppressed when $\omega_c\gg T,|\tilde{\Delta}_{pN}|$ in the limit of long-wavelength fluctuations.[25] Restricting ourselves to $q_y=0$, we then obtain

$$\begin{aligned}\bar{\chi}_{\alpha\sigma,\alpha\sigma}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}+\alpha\mathbf{Q}_{p'N},\omega_\nu)&=-\frac{T}{L_xL_y}\sum_{\mathbf{k},\omega}G_{\alpha\sigma}(\mathbf{k},\omega)G_{\overline{\alpha\sigma}}(\mathbf{k}-\alpha\mathbf{Q}_0-\mathbf{q},\omega-\omega_\nu)\\&\times I_{pN}I_{p'N}e^{-i\alpha(p-p')N(k_yb-\pi/2)},\end{aligned}\tag{A3}$$

$$\begin{aligned}\bar{\chi}_{\alpha\sigma,\overline{\alpha\sigma}}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}-\alpha\mathbf{Q}_{p'N},\omega_\nu)&=-\frac{T}{L_xL_y}\sum_{\mathbf{k},\omega}F_{\alpha\sigma}(\mathbf{k},\omega)F_{\alpha\sigma}(\mathbf{k}-\mathbf{q},\omega-\omega_\nu)\\&\times I_{pN}I_{p'N}e^{-i\alpha(p+p')N(k_yb-\pi/2)}.\end{aligned}\tag{A4}$$

Performing the sum over k_x, we obtain at $T=0$ and in the limit $|v_F q_x|,|\omega_\nu|\ll|\tilde{\Delta}_{\alpha\sigma}(k_y)|$

$$\overline{\chi}_{\alpha\sigma,\alpha\sigma}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}+\alpha\mathbf{Q}_{p'N},\omega_\nu) = I_{pN}I_{p'N}\frac{N(0)}{2N_\perp}\sum_{k_y} e^{-i\alpha(p-p')N(k_y b-\pi/2)} \times\left(\ln\frac{2E_0}{|\tilde{\Delta}_{\sigma\alpha}(k_y)|}-\frac{1}{2}-\frac{\omega_\nu^2+v_F^2q_x^2}{6|\tilde{\Delta}_{\alpha\sigma}(k_y)|^2}\right), \tag{A5}$$

$$\overline{\chi}_{\alpha\sigma,\overline{\alpha\sigma}}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}-\alpha\mathbf{Q}_{p'N},\omega_\nu) = I_{pN}I_{p'N}\frac{N(0)}{2N_\perp}\sum_{k_y} e^{-i\alpha(p+p')N(k_y b-\pi/2)} \times\left(-\frac{1}{2}+\frac{v_F^2q_x^2+\omega_\nu^2}{12|\tilde{\Delta}_{\alpha\sigma}(k_y)|^2}\right). \tag{A6}$$

Equations (A5) and (A6) are obtained by expanding to first order in $\omega_\nu^2/|\tilde{\Delta}_{\alpha\sigma}(k_y)|^2$ and $v_F^2q_x^2/|\tilde{\Delta}_{\alpha\sigma}(k_y)|^2$. This calculation is standard when evaluating the long-wavelength collective modes of a SDW system and can be found, for instance, in Ref. 25. Equations (3.29) are then obtained by summing over k_y.

It is useful to introduce the notations

$$c_{pp'}=g_2[\overline{\chi}_{+\uparrow,+\uparrow}(\mathbf{q}+\mathbf{Q}_{pN},\mathbf{q}+\mathbf{Q}_{p'N},\omega_\nu) -\overline{\chi}_{+\uparrow,-\downarrow}(\mathbf{q}+\mathbf{Q}_{pN},\mathbf{q}-\mathbf{Q}_{p'N},\omega_\nu)] =\overline{c}_{pp'}+\delta c_{pp'}, \tag{A7}$$

where $\overline{c}_{pp'}=c_{pp'}|_{\mathbf{q}=\omega_\nu=0}$ and $\delta c_{pp'}$ are deduced from Eqs. (3.29).

The gap equation can be written as a function of the static ($\omega_\nu=0$) susceptibilities $\overline{\chi}$ (or $\overline{c}$). From $\Delta_{\alpha\sigma}(\mathbf{r})=\langle O_{\alpha\sigma}(\mathbf{r})\rangle = T\Sigma_\omega F_{\alpha\sigma}(\mathbf{r},\mathbf{r},\omega)$, we deduce

$$\Delta_{pN}=I_{pN}\frac{T}{L_xL_y}\sum_{\mathbf{k},\omega,p'}\frac{e^{-i(p-p')N(k_yb-\pi/2)}}{\omega^2+\epsilon_{+\uparrow}^2(k_x)+|\tilde{\Delta}_{+\uparrow}(k_y)|^2}\tilde{\Delta}_{p'N}. \tag{A8}$$

This can be rewritten in terms of the mean-field propagators:

$$\Delta_{pN}=-I_{pN}\frac{T}{L_xL_y}\sum_{\mathbf{k},\omega,p'} e^{-i(p-p')N(k_yb-\pi/2)} \times[G_{+\uparrow}(\mathbf{k},\omega)G_{-\downarrow}(\mathbf{k}-\mathbf{Q}_0,\omega)-|F_{+\uparrow}(\mathbf{k},\omega)|^2]\tilde{\Delta}_{p'N} =\sum_{p'}\frac{1}{I_{p'N}}[\overline{\chi}_{+\uparrow,+\uparrow}(\mathbf{Q}_{pN},\mathbf{Q}_{p'N},\omega_\nu=0) -\overline{\chi}_{+\uparrow,-\downarrow}(\mathbf{Q}_{pN},-\mathbf{Q}_{p'N},\omega_\nu=0)]\tilde{\Delta}_{p'N}. \tag{A9}$$

Using the relation between Δ_{pN} and $\tilde{\Delta}_{pN}$, we obtain

$$\frac{\tilde{\Delta}_N}{(1-r^2)I_N}-\frac{r\tilde{\Delta}_{\overline{N}}}{(1-r^2)I_{\overline{N}}}=\frac{\overline{c}_{++}\tilde{\Delta}_N}{I_N},$$

$$\frac{\tilde{\Delta}_{\overline{N}}}{(1-r^2)I_{\overline{N}}}-\frac{r\tilde{\Delta}_N}{(1-r^2)I_N}=\frac{\overline{c}_{--}\tilde{\Delta}_{\overline{N}}}{I_{\overline{N}}}. \tag{A10}$$

From Eq. (A10), we deduce the relations

$$\overline{c}_{++}=\frac{1}{1+\gamma r},\quad \overline{c}_{--}=\frac{\gamma}{\gamma+r}, \tag{A11}$$

which give, by eliminating γ,

$$(1-\overline{c}_{++})(1-\overline{c}_{--})-r^2\overline{c}_{++}c_{--}=0. \tag{A12}$$

Equation (A12) is nothing but the gap equation rewritten in terms of the static mean-field susceptibilities $\overline{c}$.

In the study of the collective modes, the natural quantity to consider is not $\overline{\chi}$ but the susceptibility $\tilde{\chi}$ defined by

$$\tilde{\chi}_{\alpha\sigma,\alpha'\sigma'}(\mathbf{r},\tau;\mathbf{r}',\tau')=\langle\tilde{O}_{\alpha\sigma}(\mathbf{r},\tau)\tilde{O}^*_{\alpha'\sigma'}(\mathbf{r}',\tau')\rangle -\langle\tilde{O}_{\alpha\sigma}(\mathbf{r},\tau)\rangle\langle\tilde{O}^*_{\alpha'\sigma'}(\mathbf{r}',\tau')\rangle, \tag{A13}$$

where $\tilde{O}_\sigma(\mathbf{r},\tau)=\hat{g}(\mathbf{r})O_\sigma(\mathbf{r},\tau)$. $\tilde{\chi}$ is related to $\overline{\chi}$ by

$$\tilde{\chi}_{\alpha\sigma,\alpha'\sigma'}(\mathbf{q},\mathbf{q}',\omega_\nu) =g_2^2\overline{\chi}_{\alpha\sigma,\alpha'\sigma'}(\mathbf{q},\mathbf{q}',\omega_\nu) +g_3^2\overline{\chi}_{\overline{\alpha}\sigma,\overline{\alpha}'\sigma'}(\mathbf{q}-\alpha 4k_F,\mathbf{q}'-\alpha' 4k_F,\omega_\nu) +g_2g_3[\overline{\chi}_{\overline{\alpha}\sigma,\alpha'\sigma'}(\mathbf{q}-\alpha 4k_F,\mathbf{q}',\omega_\nu) +\overline{\chi}_{\alpha\sigma,\overline{\alpha}'\sigma'}(\mathbf{q},\mathbf{q}'-\alpha' 4k_F,\omega_\nu)] \tag{A14}$$

and satisfies

$$\tilde{\chi}_{\alpha\uparrow,\alpha'\uparrow}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}+\alpha'\mathbf{Q}_{p'N},\omega_\nu) -\tilde{\chi}_{\alpha\uparrow,\overline{\alpha}'\downarrow}(\mathbf{q}+\alpha\mathbf{Q}_{pN},\mathbf{q}-\alpha'\mathbf{Q}_{p'N},\omega_\nu) =g_2\delta_{\alpha,\alpha'}(c_{pp'}+r^2c_{\overline{pp'}})+g_3\delta_{\alpha,\overline{\alpha}'}(c_{\overline{p}p'}+c_{p\overline{p}'}). \tag{A15}$$

Equation (A15) is used in Sec. IV.

APPENDIX B: CHARGE OPERATOR ρ_{DW}

Using

$$\mathcal{G}_\alpha^{\mathrm{MF}}(\mathbf{r},\tau;\mathbf{r}',\tau')=\begin{pmatrix} G_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau') & F_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau') \\ F_{\overline{\alpha}\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau') & G_{\overline{\alpha}\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau')\end{pmatrix}, \tag{B1}$$

where G and F are the mean-field propagators (see Sec. III B), we rewrite Eq. (6.12) as

$$\rho_{\rm DW}(\mathbf{r},\tau)=-\sum_{\alpha}\int d^2r'd\tau' \times[G_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau')\tilde\eta_{\alpha\uparrow}(\mathbf{r}',\tau')F^-_{\bar\alpha\downarrow}(\mathbf{r}',\tau';\mathbf{r},\tau) + F_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau')\tilde\eta^*_{\alpha\uparrow}(\mathbf{r}',\tau')G_{\alpha\uparrow}(\mathbf{r}',\tau';\mathbf{r},\tau) + F^-_{\bar\alpha\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau')\tilde\eta_{\alpha\uparrow}(\mathbf{r}',\tau')G^-_{\bar\alpha\downarrow}(\mathbf{r}',\tau';\mathbf{r},\tau) + G^-_{\bar\alpha\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau')\tilde\eta^*_{\alpha\uparrow}(\mathbf{r}',\tau')F_{\alpha\uparrow}(\mathbf{r}',\tau';\mathbf{r},\tau)]. \tag{B2}$$

If we consider phase fluctuations only, $\eta_{\alpha\sigma}$ is given by Eq. (4.4), which gives [using Eq. (6.8)]

$$\tilde\eta_{\alpha\sigma}(\mathbf{r},\tau)=i\sum_p e^{i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}}[g_2\Delta_{pN}\varphi^{(pN)}_{\alpha\sigma}(\mathbf{r},\tau) + g_3\Delta^-_{\bar pN}\varphi^{(\overline{pN})}_{\alpha\sigma}(\mathbf{r},\tau)]. \tag{B3}$$

Defining the fluctuating phases $\tilde\varphi^{(pN)}_{\alpha\sigma}$ of the effective potential $\tilde\Delta_{\alpha\sigma}$ by

$$\tilde\Delta_{\alpha\sigma}(\mathbf{r},\tau)=\sum_p \frac{\tilde\Delta_{pN}}{I_{pN}}e^{i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}+i\tilde\varphi^{(pN)}_{\alpha\sigma}(\mathbf{r},\tau)}, \tag{B4}$$

we have

$$\tilde\eta_{\alpha\sigma}(\mathbf{r},\tau)=i\sum_p \frac{\tilde\Delta_{pN}}{I_{pN}}e^{i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}}\tilde\varphi^{(pN)}_{\alpha\sigma}(\mathbf{r},\tau). \tag{B5}$$

The relation between $\tilde\varphi^{(pN)}_{\alpha\sigma}$ and $\varphi^{(pN)}_{\alpha\sigma}$ is given by Eq. (6.15) to lowest order in phase fluctuations.

From Eqs. (B2) and (B5), we deduce

$$\rho_{\rm DW}(\tilde q)=-i\frac{T}{L_xL_y}\sum_p\frac{\tilde\Delta_{pN}}{I_{pN}}\sum_{\alpha,\tilde q'}\tilde\varphi^{(pN)}_{\alpha\uparrow}(\tilde q') \times\int d^2r\,d\tau\int d^2r'd\tau' e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_\nu\tau)+i(\mathbf{q}'\cdot\mathbf{r}'-\omega'_\nu\tau')} \times\{e^{i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}'}[G_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau')F^-_{\bar\alpha\downarrow}(\mathbf{r}',\tau';\mathbf{r},\tau) + F^-_{\bar\alpha\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau')G^-_{\bar\alpha\downarrow}(\mathbf{r}',\tau';\mathbf{r},\tau)] - e^{-i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}'}[F_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau')G_{\alpha\uparrow}(\mathbf{r}',\tau';\mathbf{r},\tau) + G^-_{\bar\alpha\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau')F_{\alpha\uparrow}(\mathbf{r}',\tau';\mathbf{r},\tau)]\}, \tag{B6}$$

where $\mathbf{q}=(q_x,q_y=0)$ and $\tilde q=(q_x,\omega_\nu)$. Using

$$\int d^2r\,d\tau\int d^2r'd\tau' e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_\nu\tau)+i(\mathbf{q}'\cdot\mathbf{r}'-\omega'_\nu\tau')+i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}'}G_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau')F^-_{\bar\alpha\downarrow}(\mathbf{r}',\tau';\mathbf{r},\tau) = \delta_{\tilde q,\tilde q'}I_{pN}\sum_{\mathbf{k},\omega}G_{\alpha\uparrow}(\mathbf{k},\omega)F^-_{\bar\alpha\downarrow}(\mathbf{k}-\alpha\mathbf{Q}_0-\mathbf{q},\omega-\omega_\nu)e^{i\alpha pN(k_yb-\pi/2)}, \tag{B7}$$

$$\int d^2r\,d\tau\int d^2r'd\tau' e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_\nu\tau)+i(\mathbf{q}'\cdot\mathbf{r}'-\omega'_\nu\tau')+i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}'}F^-_{\bar\alpha\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau')G^-_{\bar\alpha\downarrow}(\mathbf{r}',\tau';\mathbf{r},\tau) = \delta_{\tilde q,\tilde q'}I_{pN}\sum_{\mathbf{k},\omega}F^-_{\bar\alpha\downarrow}(\mathbf{k},\omega)G^-_{\bar\alpha\downarrow}(\mathbf{k}-\mathbf{q},\omega-\omega_\nu)e^{i\alpha pN(k_yb-\pi/2)}, \tag{B8}$$

$$\int d^2r\,d\tau\int d^2r'd\tau' e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_\nu\tau)+i(\mathbf{q}'\cdot\mathbf{r}'-\omega'_\nu\tau')-i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}'}F_{\alpha\uparrow}(\mathbf{r},\tau;\mathbf{r}',\tau')G_{\alpha\uparrow}(\mathbf{r}',\tau';\mathbf{r},\tau) = \delta_{\tilde q,\tilde q'}I_{pN}\sum_{\mathbf{k},\omega}F_{\alpha\uparrow}(\mathbf{k},\omega)G_{\alpha\uparrow}(\mathbf{k}-\mathbf{q},\omega-\omega_\nu)e^{-i\alpha pN(k_yb-\pi/2)}, \tag{B9}$$

$$\int d^2r\,d\tau\int d^2r'd\tau' e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_\nu\tau)+i(\mathbf{q}'\cdot\mathbf{r}'-\omega'_\nu\tau')-i\alpha\mathbf{Q}_{pN}\cdot\mathbf{r}'}G^-_{\bar\alpha\downarrow}(\mathbf{r},\tau;\mathbf{r}',\tau')F_{\alpha\uparrow}(\mathbf{r}',\tau';\mathbf{r},\tau) = \delta_{\tilde q,\tilde q'}I_{pN}\sum_{\mathbf{k},\omega}G^-_{\bar\alpha\downarrow}(\mathbf{k},\omega)F_{\alpha\uparrow}(\mathbf{k}+\alpha\mathbf{Q}_0-\mathbf{q},\omega-\omega_\nu)e^{-i\alpha pN(k_yb-\pi/2)}, \tag{B10}$$

we obtain

$$\rho_{\rm DW}(\tilde q)=-i\sum_{\alpha,p}\tilde\Delta_{pN}\tilde\varphi^{(pN)}_{\alpha\uparrow}(\tilde q)\frac{T}{L_xL_y}\sum_{\mathbf{k},\omega}\{e^{i\alpha pN(k_yb-\pi/2)}[G_{\alpha\uparrow}(\mathbf{k},\omega)F^-_{\bar\alpha\downarrow}(\mathbf{k}-\alpha\mathbf{Q}_0-\mathbf{q},\omega-\omega_\nu)+F^-_{\bar\alpha\downarrow}(\mathbf{k},\omega)G^-_{\bar\alpha\downarrow}(\mathbf{k}-\mathbf{q},\omega-\omega_\nu)] - e^{-i\alpha pN(k_yb-\pi/2)}[F_{\alpha\uparrow}(\mathbf{k},\omega)G_{\alpha\uparrow}(\mathbf{k}-\mathbf{q},\omega-\omega_\nu)+G^-_{\bar\alpha\downarrow}(\mathbf{k},\omega)F_{\alpha\uparrow}(\mathbf{k}+\alpha\mathbf{Q}_0-\mathbf{q},\omega-\omega_\nu)]\}. \tag{B11}$$

To lowest order in v_Fq_x and ω_ν, we have (for $q_y=0$)

$$\frac{T}{bL_x}\sum_{k_x,\omega} G_{\alpha\sigma}(\mathbf{k},\omega)F_{\overline{\alpha\sigma}}(\mathbf{k}-\alpha\mathbf{Q}_0-\mathbf{q},\omega-\omega_\nu) = -N(0)\frac{\tilde{\Delta}^*_{\alpha\uparrow}(k_y)}{8|\tilde{\Delta}_{\alpha\uparrow}(k_y)|^2}(i\omega_\nu+\alpha v_F q_x), \quad \text{(B12)}$$

$$\frac{T}{bL_x}\sum_{k_x,\omega} F_{\overline{\alpha\sigma}}(\mathbf{k},\omega)G_{\overline{\alpha\sigma}}(\mathbf{k}-\mathbf{q},\omega-\omega_\nu) = -N(0)\frac{\tilde{\Delta}^*_{\alpha\uparrow}(k_y)}{8|\tilde{\Delta}_{\alpha\uparrow}(k_y)|^2}(-i\omega_\nu+\alpha v_F q_x). \quad \text{(B13)}$$

This leads to

$$\rho_{\rm DW}(\tilde{q}) = \frac{iN(0)}{4N_\perp}\sum_{p,\alpha,k_y}\tilde{\Delta}_{pN}\alpha v_F q_x e^{i\alpha pN(k_y b-\pi/2)} \times\frac{\tilde{\Delta}^*_{\alpha\uparrow}(k_y)}{|\tilde{\Delta}_{\alpha\uparrow}(k_y)|^2}[\tilde{\varphi}^{(pN)}_{\alpha\uparrow}(\tilde{q})-\tilde{\varphi}^{(\bar{p}N)}_{\alpha\uparrow}(\tilde{q})]. \quad \text{(B14)}$$

Performing the sum over k_y, we finally obtain

$$\rho_{\rm DW}(\tilde{q}) = i\frac{q_x}{2\pi b}[\tilde{\varphi}^{(pN)}_{+\uparrow}(\tilde{q})-\tilde{\varphi}^{(pN)}_{-\uparrow}(\tilde{q})], \quad \text{(B15)}$$

which yields Eq. (6.13).

APPENDIX C: MEAN-FIELD SUSCEPTIBILITIES IN THE HELICOIDAL PHASE

In the helicoidal phase, the mean-field susceptibilities are given by

$$\overline{\chi}_{\alpha\sigma,\alpha\sigma}(\mathbf{q}+\alpha\mathbf{Q}_{p(\alpha,\sigma)N},\mathbf{q}+\alpha\mathbf{Q}_{p(\alpha,\sigma)N},\omega_\nu) = I^2_{p(\alpha,\sigma)N}\frac{N(0)}{2}\left[\ln\frac{2E_0}{|\tilde{\Delta}_{\alpha\sigma}|}-\frac{1}{2}-\frac{\omega_\nu^2+v_F^2q_x^2}{6\tilde{\Delta}^2_{\alpha\sigma}}\right], \quad \text{(C1)}$$

$$\overline{\chi}_{\alpha\sigma,\overline{\alpha\sigma}}(\mathbf{q}+\alpha\mathbf{Q}_{p(\alpha,\sigma)N},\mathbf{q}-\alpha\mathbf{Q}_{p(\alpha,\sigma)N},\omega_\nu) = I^2_{p(\alpha,\sigma)N}\frac{N(0)}{2}\left[-\frac{1}{2}+\frac{\omega_\nu^2+v_F^2q_x^2}{12\tilde{\Delta}^2_{\alpha\sigma}}\right], \quad \text{(C2)}$$

for $q_y=0$ and $|\omega_\nu|,v_F|q_x|\ll|\tilde{\Delta}_{\alpha\sigma}|$. Introducing the notations

$$c_{++} = g_2[\overline{\chi}_{+\uparrow,+\uparrow}(\mathbf{q}+\mathbf{Q}_N,\mathbf{q}+\mathbf{Q}_N,\omega_\nu) - \overline{\chi}_{+\uparrow,-\downarrow}(\mathbf{q}+\mathbf{Q}_N,\mathbf{q}-\mathbf{Q}_N,\omega_\nu)],$$

$$c_{--} = g_2[\overline{\chi}_{-\uparrow,-\uparrow}(\mathbf{q}-\mathbf{Q}_{\bar{N}},\mathbf{q}-\mathbf{Q}_{\bar{N}},\omega_\nu) - \overline{\chi}_{-\uparrow,+\downarrow}(\mathbf{q}-\mathbf{Q}_{\bar{N}},\mathbf{q}+\mathbf{Q}_{\bar{N}},\omega_\nu)], \quad \text{(C3)}$$

the gap equation reads

$$(1-\overline{c}_{++})(1-\overline{c}_{--})-r^2\overline{c}_{++}\overline{c}_{--}=0, \quad \text{(C4)}$$

where $\overline{c}_{++}=1/(1+\gamma r)$ and $\overline{c}_{--}=\gamma/(\gamma+r)$ ($\overline{c}_{pp} = c_{pp}|_{\tilde{q}=0}$).

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*Present address: Laboratoire de Physique des Solides, Université Paris-Sud, 91405 Orsay, France.

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