Chapter 2

Symmetries *(Last version: 10 September 2019)*

Contents

This chapter is concerned with a discussion of symmetries in quantum many-body systems. After a general introduction to symmetries, invariances and conservation laws (Sec. 2.1), we consider a number of standard symmetry transformations: geometric space transformations (e.g. parity or space rotation), time translation, Galilean transformation, time reversal and gauge transformation (Sec. 2.2). In section 2.3, we briefly review Noether's theorem in classical field theory and show how the invariance of a quantum system in a symmetry transformation constrains the correlation functions, implying in particular relations, known as Ward identities, between the Green functions or the 1PI/2PI vertices.

2.1 Symmetries, invariances and conservation laws

2.1.1 Symmetry transformations in quantum mechanics

Physical states are represented by rays in the Hilbert space. A ray is a set of normalized vectors which differ only by a phase factor: $|\Psi\rangle$ and $|\Psi'\rangle$ belong to the same ray if $|\Psi'\rangle = e^{i\alpha}|\Psi\rangle$ (α real). For a system in the state represented by the ray $\mathcal{R}$, the probability to find it in the state represented by the ray $\mathcal{R}_n$ is

$$P(\mathcal{R} \to \mathcal{R}_n) = |\langle\Psi_n|\Psi\rangle|^2, \tag{2.1}$$

where the vectors $|\Psi\rangle$ and $|\Psi_n\rangle$ belong to the rays $\mathcal{R}$ and $\mathcal{R}_n$, respectively.

Let us consider two observers $\mathcal{O}$ and $\mathcal{O}'$ who look at the same system.[1] They of course find the same result, in their own reference frame, for a given experiment. We shall say that $\mathcal{O}$ and $\mathcal{O}'$ are related by a symmetry transformation $\mathcal{T}$. When $\mathcal{O}$ sees the system in a state represented by the ray $\mathcal{R}$ ($\mathcal{R}_n$), the equivalent observer $\mathcal{O}'$ sees the system in a state represented by the ray $\mathcal{R}'$ ($\mathcal{R}'_n$). The two observers find the same probabilities $P(\mathcal{R} \to \mathcal{R}_n) = P(\mathcal{R}' \to \mathcal{R}'_n)$, i.e.

$$|\langle\Psi|\Psi_n\rangle|^2 = |\langle\Psi'|\Psi'_n\rangle|^2. \tag{2.2}$$

It is customary to call $\mathcal{T}$ a symmetry transformation regardless of whether the physical system itself possesses the symmetry. The conditions for the system to be invariant under a symmetry transformation will be discussed in section 2.1.5. Note that a symmetry transformation can be time dependent.

2.1.2 Wigner's theorem – Group structure

The Wigner theorem states that for any transformation $\mathcal{T} : \mathcal{R} \mapsto \mathcal{R}' = \mathcal{T}(\mathcal{R})$ which conserves the probability, $|\langle\Psi|\Phi\rangle| = |\langle\Psi'|\Phi'\rangle|$, there is an operator $\hat{U} \equiv \hat{U}(\mathcal{T})$ such that $|\Psi'\rangle = \hat{U}|\Psi\rangle$, which is either unitary or antiunitary.[2] $\hat{U}(\mathcal{T})$ is not uniquely defined: if $\hat{U}$ is an operator corresponding to the transformation $\mathcal{T}$ so is $e^{i\alpha}\hat{U}$ for any real constant α. The operator $\hat{U}$ corresponding to time reversal symmetry is antiunitary. In all other cases known in physics, $\hat{U}$ is unitary. We shall discussed time reversal symmetry in section 2.2.6 and assume otherwise $\hat{U}$ to be unitary. The identity transformation being unitary, transformations that can be continuously connected to it are necessarily unitary. These are discussed in the next section.

The set of symmetry transformations $\mathcal{T}$ has the properties of a group. The product of two symmetry transformations, $\mathcal{T}_1 : \mathcal{R} \mapsto \mathcal{T}_1(\mathcal{R})$ and $\mathcal{T}_2 : \mathcal{R} \mapsto \mathcal{T}_2(\mathcal{R})$, is defined by $\mathcal{T}_1\mathcal{T}_2 : \mathcal{R} \mapsto \mathcal{T}_1(\mathcal{T}_2(\mathcal{R}))$. The identity transformation leaves rays unchanged, and any transformation $\mathcal{T} : \mathcal{R} \mapsto \mathcal{R}'$ has an inverse defined by $\mathcal{T}^{-1} : \mathcal{R}' \mapsto \mathcal{R}$. The operators $\hat{U}(\mathcal{T})$ corresponding to the symmetry transformations $\mathcal{T}$ are expected to have a similar group structure. Since the $\hat{U}(\mathcal{T})$'s act on states of the Hilbert space, and not on rays, the requirement that $\hat{U}(\mathcal{T}_1\mathcal{T}_2)|\Psi\rangle$ and $\hat{U}(\mathcal{T}_1)\hat{U}(\mathcal{T}_2)|\Psi\rangle$ belong to the same ray leads to[3]

$$\hat{U}(\mathcal{T}_1)\hat{U}(\mathcal{T}_2) = e^{i\alpha(\mathcal{T}_1,\mathcal{T}_2)}\hat{U}(\mathcal{T}_1\mathcal{T}_2). \tag{2.3}$$

[1] This is the so-called passive point of view. Alternatively, one can adopt the active point of view with a single observer $\mathcal{O}$ and two systems $\mathcal{S}$ and $\mathcal{S}'$ [4].

[2] A linear operator $\hat{U}$ satisfies $\hat{U}(\alpha|\Psi\rangle + \beta|\Phi\rangle) = \alpha\hat{U}|\Psi\rangle + \beta\hat{U}\beta|\Phi\rangle$. $\hat{U}$ is unitary if it is invertible with $\hat{U}^{-1} = \hat{U}^\dagger$. Antiunitary operators are defined in Sec. 2.2.6.

[3] This equation follows from $\hat{U}(\mathcal{T}_1)\hat{U}(\mathcal{T}_2)|\Psi\rangle = e^{i\alpha(\mathcal{T}_1,\mathcal{T}_2)}\hat{U}(\mathcal{T}_1\mathcal{T}_2)|\Psi\rangle$ where the phase $\alpha(\mathcal{T}_1,\mathcal{T}_2)$ can be shown to be independent of $|\Psi\rangle$ [4, 5].

Thus, the operators $\hat{U}(\mathcal{T})$ provide a representation up to a phase (called a projective representation) of the group of symmetry transformations $\mathcal{T}$. It turns out that when the phase $\alpha(\mathcal{T}_1,\mathcal{T}_2)\neq 0$ cannot be eliminated by a (trivial) redefinition of $\hat{U}(\mathcal{T})$, the symmetry group can always be enlarged in such a way that its representations can all be defined as non projective. Assuming that this has been done, we take $\alpha(\mathcal{T}_1,\mathcal{T}_2)=0$ in the following.

The Wigner operator depends on time if the symmetry transformation does. Unless necessary to avoid confusion, we will not explicitly indicate the time dependence of the Wigner operator, the Hamiltonian, and the state vectors of the Hilbert space.

2.1.3 Continuous symmetries

In this section, we consider transformations described by a finite set $\theta=\{\theta_a\}$ of real parameters and continuously connected to the identity. Such transformations form a Lie group. The multiplication law takes the form

$$\mathcal{T}(\theta)\mathcal{T}(\theta')=\mathcal{T}(f(\theta,\theta')), \tag{2.4}$$

where $f_a(\theta,\theta')$ is a function of θ and θ'. If we choose $\theta_a=0$ as the coordinates of the identity transformation, we must have

$$f_a(\theta,0)=f_a(0,\theta)=\theta_a. \tag{2.5}$$

If the operators $\hat{U}(\theta)\equiv\hat{U}(\mathcal{T}(\theta))$ form a non-projective representation of the group of symmetry transformations, they satisfy

$$\hat{U}(\theta)\hat{U}(\theta')=\hat{U}(f(\theta,\theta')). \tag{2.6}$$

This equation leads to important consequences when expanded in powers of θ_a and θ'_a. In the neighborhood of the identity,

$$\hat{U}(\theta)=1+i\sum_a\theta_a\hat{t}_a+\frac{1}{2}\sum_{a,b}\theta_a\theta_b\hat{t}_{ab}+\cdots \tag{2.7}$$

where $\hat{t}_a$ and $\hat{t}_{ab}$ are Hermitian operators (as required by the unitarity of $\hat{U}(\theta)$). We have

$$f_a(\theta,\theta')=\theta_a+\theta'_a+\sum_c f_{abc}\theta_b\theta'_c+\cdots \tag{2.8}$$

since terms proportional to $\theta_b\theta_c$ or $\theta'_b\theta'_c$ are forbidden by (2.5). By inserting (2.7) and (2.8) into (2.6), we conclude that the transformation law (2.6) holds at order $\theta_a\theta'_b$ only if

$$\hat{t}_{ab}=-\hat{t}_a\hat{t}_b-i\sum_c f_{cab}\hat{t}_c. \tag{2.9}$$

To derive equation (2.9), we have used the fact that $\hat{t}_{ab}=\hat{t}_{ba}$ is symmetric with respect to a and b, since it comes from the second-order derivative of $\hat{U}(\theta)$. The symmetry of the lhs of (2.9) with respect to a and b then yields

$$[\hat{t}_a,\hat{t}_b]=i\sum_c(f_{cba}-f_{cab})\hat{t}_c=i\sum_c C_{cab}\hat{t}_c. \tag{2.10}$$

Symmetry transformation	Wigner operator $\hat{U}$	Generator
Space translation	$\exp(-i\mathbf{a}\cdot\hat{\mathbf{P}})$	momentum $\hat{\mathbf{P}}$
Space rotation	$\exp(-i\theta\mathbf{n}\cdot\hat{\mathbf{J}})$	angular momentum $\hat{\mathbf{J}}=\hat{\mathbf{L}}+\hat{\mathbf{S}}$
Time translation	$\hat{T}(t+\tau,t)$	Hamiltonian $\hat{H}$
Galilean transformation	$\exp[i(\hat{\mathbf{P}}t-M\hat{\mathbf{R}}).\mathbf{v}]$	$-\hat{\mathbf{P}}t+M\hat{\mathbf{R}}$
U(1) transformation	$\exp(i\Lambda\hat{N})$	Particle number $\hat{N}$

Table 2.1: Continuous symmetry transformations. (M is the total mass of the system and $\hat{\mathbf{R}}$ the position of its center of mass.)

The operators $\hat{t}_a$ are the generators of the group and the real constants $C_{cab}=f_{cba}-f_{cab}$ the structure constants. Equation (2.9) means that for a given function $f(\theta,\theta')$, the second-order terms in the expansion of $\hat{U}(\theta)$ are completely determined by the first-order terms, i.e. the generators $\hat{t}_a$. The set of commutation relations (2.10) is known as a Lie algebra. It can be shown that these relations are sufficient to obtain the complete power series of $\hat{U}(\theta)$ and uniquely determine the operator $\hat{U}(\theta)$ (at least in the neighborhood of $\theta_a=0$) [5].

When the function $f_a(\theta,\theta')=\theta_a+\theta'_a$ is simply additive, the operator $\hat{U}(\theta)$ can be explicitly obtained. In this case $f_{abc}=0$ and the generators commute (Abelian group): $[\hat{t}_a,\hat{t}_b]=0$. Writing $\hat{U}(\theta)$ as an infinite number of infinitesimal transformations, we obtain

$$\hat{U}(\theta)=\lim_{N\to\infty}\left[\hat{U}\left(\frac{\theta}{N}\right)\right]^N=\lim_{N\to\infty}\left(1+i\sum_a\frac{\theta_a}{N}\hat{t}_a\right)^N=\exp\Bigl(i\sum_a\theta_a\hat{t}_a\Bigr). \tag{2.11}$$

For example, rotations of angle θ about a given axis $\mathbf{n}$ form an Abelian group and the corresponding Wigner operator can be cast in the form (2.11) (Sec. 2.2.3).

Continuous symmetry transformations play a very important role. Examples are given in table 2.1.

2.1.4 Transformation of observables

In order to ensure that the transformation $\mathcal{T}$ conserves the physical properties of the system, an operator $\hat{A}$ should transform as

$$\hat{A}'=\hat{U}\hat{A}\hat{U}^\dagger \tag{2.12}$$

so that $\langle\Psi'|\hat{A}'|\Phi'\rangle=\langle\Psi|\hat{A}|\Phi\rangle$.

However, sometimes the operator $\hat{A}'$ is not used by the observer $\mathcal{O}'$ to make measurements on the system. Let us consider for instance the quantum mechanics of a single particle and call $\hat{\mathbf{r}}$ the operator for the distance between the particle and the origin of the reference frame used by $\mathcal{O}$. $\hat{U}\hat{\mathbf{r}}\hat{U}^\dagger$ is the operator used by $\mathcal{O}'$ for the same distance. But $\mathcal{O}'$ is rather interested in the position of the particle as measured with respect to its own reference frame and will therefore used the (non-transformed) operator $\hat{\mathbf{r}}$.[4]

[4]Note that the same operator, $\hat{\mathbf{r}}$, is used by $\mathcal{O}$ and $\mathcal{O}'$ to describe the distance of a particle to the origin of their respective reference frames.

It may be worth considering a particular example. Let us suppose that $\mathcal{O}'$ uses a reference frame translated by $-\mathbf{a}$ with respect to that used by $\mathcal{O}$. In the coordinate basis, $\Psi'(\mathbf{r}) = \Psi(\mathbf{r}-\mathbf{a})$ (see Sec. 2.2.2). The particle position as seen by $\mathcal{O}'$ is then $\langle\Psi'|\hat{\mathbf{r}}|\Psi'\rangle = \int d^dr|\Psi'(\mathbf{r})|^2\mathbf{r} = \int d^dr|\Psi(\mathbf{r})|^2(\mathbf{r}+\mathbf{a}) = \langle\Psi|\hat{\mathbf{r}}|\Psi\rangle + \mathbf{a}$.

Thus, in going from $\mathcal{O}$ to $\mathcal{O}'$, we shall not transform the operators corresponding to coordinates, momenta and spins of the particles. In some cases, however, it is necessary to transform the operator as in (2.12). Consider for instance the case of an external vector potential $\hat{\mathbf{A}}(\mathbf{r})$ in ordinary quantum mechanics. Assuming that $\mathcal{O}'$ uses a reference framed rotated wrt that of $\mathcal{O}$, choosing $\hat{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{A}}'(\mathbf{r})$ amounts to having rotated the external field as well.

Let us go back to the case where the observable is not transformed. One can see the symmetry transformation as changing the operators rather than the states, since $\langle\Psi'|\hat{A}|\Phi'\rangle = \langle\Psi|\hat{U}^\dagger\hat{A}\hat{U}|\Phi\rangle$. In the following, we will sometimes take this point of view. A given symmetry transformation is then defined by the "transformation law" of a set of irreducible operators, e.g. the position, momentum and spin operators for a single particle.

Let us finally discuss the transformation law of the Hamiltonian. The transformed Hamiltonian $\hat{H}'$ determines the Schrödinger equation satisfied by the transformed vectors. From $i\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle$ and

$$i\partial_t(\hat{U}|\Psi\rangle) = i(\partial_t\hat{U})|\Psi\rangle + i\hat{U}\partial_t|\Psi\rangle = [i(\partial_t\hat{U})\hat{U}^\dagger + \hat{U}\hat{H}\hat{U}^\dagger]\hat{U}|\Psi\rangle, \tag{2.13}$$

we deduce

$$\hat{H}' = \hat{U}\hat{H}\hat{U}^\dagger + i(\partial_t\hat{U})\hat{U}^\dagger \tag{2.14}$$

and

$$\langle\Psi'|\hat{H}'|\Psi'\rangle = \langle\Psi|\hat{H}|\Psi\rangle + i\langle\Psi|\hat{U}^\dagger(\partial_t\hat{U})|\Psi\rangle. \tag{2.15}$$

The expectation value of the energy is not the same for the two observers when the transformation is time dependent (see, e.g. Sec. 2.2.5).

2.1.5 Invariance – Conservation laws

A system is invariant under the symmetry transformation $\mathcal{T}$ if $|\Psi\rangle$ and $\hat{U}|\Psi\rangle$ satisfy the same Schrödinger equation, i.e. $\hat{H}' = \hat{H}$. Equation (2.14) then implies

$$[\hat{U}, \hat{H}] + i\partial_t\hat{U} = 0. \tag{2.16}$$

If the operator $\hat{U}$ is time independent, the invariance condition (2.16) reduces to $[\hat{U}, \hat{H}] = 0$. When $\hat{H}' = \hat{H}$, any vector $|\Psi\rangle$ solution of the Schrödinger equation $\hat{H}|\Psi\rangle = i\partial_t|\Psi\rangle$, and thus describing a possible physical state of the system as seen by $\mathcal{O}$, also corresponds to a possible physical state of the system as seen by $\mathcal{O}'$. The physical state described by $|\Psi\rangle$ looks the same subjectively to $\mathcal{O}$ and $\mathcal{O}'$, but obviously refers to objectively different physical situations for the system: The laws of physics are invariant under the symmetry transformation $\mathcal{O} \to \mathcal{O}'$.

The fact that $|\psi(t)\rangle$ and $|\psi'(t)\rangle = \hat{U}(t)|\psi(t)\rangle$ share the same time evolution can also be written as $|\psi'(t)\rangle = \hat{T}(t,t_0)|\psi'(t_0)\rangle$, where the evolution operator $\hat{T}$ is defined by $|\psi(t)\rangle = \hat{T}(t,t_0)|\psi(t_0)\rangle$. Since this is true for any state $|\psi(t_0)\rangle$, this implies

$$\hat{U}(t)\hat{T}(t,t_0) = \hat{T}(t,t_0)\hat{U}(t_0), \tag{2.17}$$

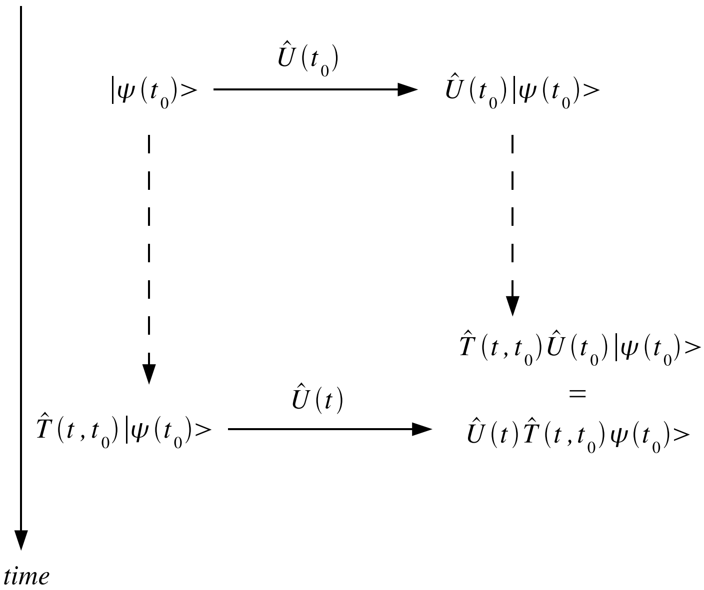

Figure 2.1: Schematic representation of the invariance under the symmetry transformation $\hat{U}(t)$.

which is a mere reformulation of (2.16). Thus to go from $|\psi(t_0)\rangle$ to $|\psi'(t)\rangle$, one can either first obtain $|\psi'(t_0)\rangle$ using $\hat{U}(t_0)$ and then let the state evolve in time with $\hat{T}(t,t_0)$, or first let the state $|\psi(t_0)\rangle$ evolve into $|\psi(t)\rangle$ and then obtain $|\psi'(t)\rangle$ with $\hat{U}(t)$ (Fig. 2.1).

So far we have used the Schrödinger picture where the symmetry transformation affects the vectors but not the operators such as position, momentum and spin. In the Heisenberg picture, the wave functions are time independent, taken to be equal to their values at t_0, while the operators become time dependent: $\hat{A}_H(t) = \hat{T}^\dagger(t,t_0)\hat{A}\hat{T}(t,t_0)$. Since in general $\hat{T}$ and $\hat{T}'$ (the evolution operator used by $\mathcal{O}'$) differ, both the vectors and the operators change when passing from $\mathcal{O}$ to $\mathcal{O}'$. However, when the system is invariant under the symmetry transformation, $\hat{H} = \hat{H}'$ and therefore $\hat{T}(t,t_0) = \hat{T}'(t,t_0)$. It follows that the operator $\hat{A}_H(t)$ in the Heisenberg picture is the same for the two observers $\mathcal{O}$ and $\mathcal{O}'$.

Equations (2.16) and (2.17) take a very simple form in the Heisenberg picture,

$$\frac{d\hat{U}_H}{dt} = 0. \tag{2.18}$$

Thus the system is invariant under the symmetry transformation $\mathcal{T}$ if the corresponding Wigner operator $\hat{U}_H$ is a constant of motion: Invariance implies conservation laws. For a continuous symmetry, a necessary and sufficient condition is that each generator of the group is a constant of motion. Considering (2.18) in the case of an infinitesimal transformation, $\hat{U}_H(\theta) \simeq 1 + i\sum_a \theta_a \hat{t}_{aH}$ (see Sec. 2.1.3), we obtain

$$\frac{d\hat{t}_{aH}}{dt} = 0, \tag{2.19}$$

where the $\hat{t}_{aH}$'s are the generators in the Heisenberg picture. If the Wigner operator is time independent, equation (2.19) reduces to $[\hat{H}, \hat{t}_a] = 0$.

The consequences of an invariance in a symmetry transformation are further discussed in sections 2.2 and 2.3, and in chapter 3.

2.2 Examples of symmetry transformations

2.2.1 Parity

We consider the transformation $\mathcal{T}$ where the two observers use reference frames with opposite directions of the coordinate axis: $x' = -x$, $y' = -y$, etc. This is an example of a discrete (as opposed to continuous) transformation. Since two successive parity transformations reduce to the identity transformation, $\mathcal{T}$ and $\mathcal{T}^2 = 1$ form a group.

Let us first discuss the case of a single particle. By analogy with the parity transformation in classical mechanics, $\mathbf{r}' = -\mathbf{r}$ and $\mathbf{p}' = -\mathbf{p}$, we determine the Wigner operator $\hat U$ from

$$\begin{aligned} \langle\Psi|\hat U^\dagger \hat{\mathbf{r}} \hat U|\Psi\rangle &= -\langle\Psi|\hat{\mathbf{r}}|\Psi\rangle, \\ \langle\Psi|\hat U^\dagger \hat{\mathbf{p}} \hat U|\Psi\rangle &= -\langle\Psi|\hat{\mathbf{p}}|\Psi\rangle. \end{aligned} \tag{2.20}$$

Since these relations hold for any vector $|\Psi\rangle$, they imply

$$\begin{aligned} \hat U^\dagger \hat{\mathbf{r}} \hat U &= \mathcal{U}(\hat{\mathbf{r}}), \\ \hat U^\dagger \hat{\mathbf{p}} \hat U &= -\hat{\mathbf{p}}. \end{aligned} \tag{2.21}$$

The spin of the particle should transform as the orbital angular momentum $\hat{\mathbf{r}} \times \hat{\mathbf{p}}$ and is invariant under a parity transformation; we ignore it in the following. To make the discussion slightly more general and suitable to other space transformations, we have introduced the function $\mathcal{U}(\mathbf{r}) = -\mathbf{r}$. Note that equations (2.21) defines $\hat U$ only up to a phase factor. For an eigenstate $|\mathbf{r}\rangle$ of the position operator $\hat{\mathbf{r}}$, we obtain

$$\hat{\mathbf{r}} \hat U|\mathbf{r}\rangle = \hat U \mathcal{U}(\hat{\mathbf{r}})|\mathbf{r}\rangle = \mathcal{U}(\mathbf{r}) \hat U|\mathbf{r}\rangle. \tag{2.22}$$

$|\mathcal{U}(\mathbf{r})\rangle$ and $\hat U|\mathbf{r}\rangle$ being both eigenstates of $\hat{\mathbf{r}}$ with eigenvalue $\mathcal{U}(\mathbf{r})$, they should coincide up to a phase, $\hat U|\mathbf{r}\rangle = e^{i\alpha(\mathbf{r})}|\mathcal{U}(\mathbf{r})\rangle$. Choosing $\alpha(\mathbf{r}) = 0$, we obtain

$$\hat U|\mathbf{r}\rangle = |\mathcal{U}(\mathbf{r})\rangle. \tag{2.23}$$

Since $\hat U^\dagger = \hat U^{-1}$, we also have $\hat U^\dagger|\mathbf{r}\rangle = |\mathcal{U}^{-1}(\mathbf{r})\rangle$. Equation (2.23) completely determines $\hat U$. This definition ensures that $\{\hat U, \hat U^2 = 1\}$ form a group and is compatible with the transformation law of $\mathbf{p}$ [Eq. (2.21)]:

$$\hat U^{(\dagger)}|\mathbf{p}\rangle = \int d^d r\, \hat U^{(\dagger)}|\mathbf{r}\rangle\langle\mathbf{r}|\mathbf{p}\rangle = \int d^d r |-\mathbf{r}\rangle\langle\mathbf{r}|\mathbf{p}\rangle = |-\mathbf{p}\rangle \tag{2.24}$$

(we have used $\langle\mathbf{r}|\mathbf{p}\rangle = \langle -\mathbf{r}|-\mathbf{p}\rangle$) and therefore $\hat U^\dagger \hat{\mathbf{p}} \hat U = -\hat{\mathbf{p}}$. The wave function $\Psi(\mathbf{r}) = \langle\mathbf{r}|\Psi\rangle$ transforms into

$$\Psi'(\mathbf{r}) = \langle\mathbf{r}|\hat U|\Psi\rangle = \int d^d r' \langle\mathbf{r}|\hat U|\mathbf{r}'\rangle\langle\mathbf{r}'|\Psi\rangle = \int d^d r' \langle\mathbf{r}|\mathcal{U}(\mathbf{r}')\rangle\Psi(\mathbf{r}') = \Psi(\mathcal{U}^{-1}(\mathbf{r})). \tag{2.25}$$

Invariance under a parity transformation means that the parity is a good quantum number: $[\hat U, \hat H] = 0$ and $d\hat U_H/dt = 0$.

Many-particle system

For a many-particle system, we define the Wigner operator of the parity transformation by

$$\hat{U}|\mathbf{r}_1,\cdots,\mathbf{r}_N\} = |\mathcal{U}(\mathbf{r}_1),\cdots,\mathcal{U}(\mathbf{r}_N)\}, \tag{2.26}$$

where $|\mathbf{r}_1,\cdots,\mathbf{r}_N\} = \hat{\psi}^\dagger(\mathbf{r}_1)\cdots\hat{\psi}^\dagger(\mathbf{r}_N)|\text{vac}\rangle$, with $|\text{vac}\rangle$ the vacuum of particles (Sec. 1.2.1). By evaluating

$$\begin{aligned}\hat{U}^\dagger\hat{\psi}^\dagger(\mathbf{r})\hat{U}|\mathbf{r}_1,\cdots,\mathbf{r}_N\} &= \hat{U}^\dagger\hat{\psi}^\dagger(\mathbf{r})|\mathcal{U}(\mathbf{r}_1),\cdots,\mathcal{U}(\mathbf{r}_N)\} \\ &= \hat{U}^\dagger|\mathbf{r},\mathcal{U}(\mathbf{r}_1),\cdots,\mathcal{U}(\mathbf{r}_N)\} \\ &= |\mathcal{U}^{-1}(\mathbf{r}),\mathbf{r}_1,\cdots,\mathbf{r}_N\},\end{aligned} \tag{2.27}$$

we conclude that the operators $\hat{\psi}^{(\dagger)}(\mathbf{r})$ transform as

$$\hat{U}^\dagger\hat{\psi}^{(\dagger)}(\mathbf{r})\hat{U} = \hat{\psi}^{(\dagger)}(\mathcal{U}^{-1}(\mathbf{r})). \tag{2.28}$$

Equations (2.25) and (2.28), which follow from (2.23), are general and do not depend on the space transformation $\mathcal{U}(\mathbf{r})$. The operators $\hat{\psi}(\mathbf{r})$ and the wave function $\Psi(\mathbf{r})$ transform analogously but in an opposite way wrt the transformation of the eigenstates $|\mathbf{r}\rangle$ of the position operator $\hat{\mathbf{r}}$. The case $\mathcal{U}(\mathbf{r}) = -\mathbf{r}$ is special in this respect since $\mathcal{U}^{-1} = \mathcal{U}$.

Equation (2.28) has an important consequence for the parity transformation, $\mathcal{U}(\mathbf{r}) = -\mathbf{r}$. A local operator $\hat{A}(\mathbf{r})$, which can be written as a product of the fields $\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r})$ and their gradients, transforms as $\hat{U}^\dagger\hat{A}(\mathbf{r})\hat{U} = \epsilon_A^P\hat{A}(-\mathbf{r})$ where $\epsilon_A^P = \pm 1$ is the signature of the operator $\hat{A}$ under the parity transformation. $\epsilon_A^P = 1$ (-1) if $\hat{A}(\mathbf{r})$ involves an even (odd) number of derivative terms $\boldsymbol{\nabla}\hat{\psi}(\mathbf{r})$ or $\boldsymbol{\nabla}\hat{\psi}^\dagger(\mathbf{r})$. For instance, the particle density operator $\hat{n}(\mathbf{r}) = \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})$ is even under a parity transformation, whereas the current operator $\hat{\mathbf{j}}(\mathbf{r}) = -\frac{i}{2m}(\hat{\psi}^\dagger(\mathbf{r})\boldsymbol{\nabla}\hat{\psi}(\mathbf{r}) - \text{h.c.})$ is odd:[5]

$$\begin{aligned}\hat{U}^\dagger\hat{n}(\mathbf{r})\hat{U} &= \hat{n}(-\mathbf{r}) \qquad (\epsilon_n^P = 1), \\ \hat{U}^\dagger\hat{\mathbf{j}}(\mathbf{r})\hat{U} &= -\hat{\mathbf{j}}(-\mathbf{r}) \qquad (\epsilon_j^P = -1).\end{aligned} \tag{2.29}$$

From these results we can deduce how correlation functions transform in a parity transformation. If the system is invariant under parity, $[\hat{H},\hat{U}] = 0$ and $\hat{H}' = \hat{H}$, one finds

$$\hat{U}^\dagger\hat{\psi}(\mathbf{r},t)\hat{U} = e^{i\hat{H}t}\hat{U}^\dagger\hat{\psi}(\mathbf{r})\hat{U}e^{-i\hat{H}t} = \hat{\psi}(-\mathbf{r},t), \tag{2.30}$$

where $\hat{\psi}(\mathbf{r},t) = e^{i\hat{H}t}\hat{\psi}(\mathbf{r})e^{-i\hat{H}t}$ is the field operator in the Heisenberg representation. Furthermore, since $\mathcal{O}$ and $\mathcal{O}'$ measure the same correlation functions in a parity-invariant system,

$$\begin{aligned}\langle 0|\hat{\psi}(\mathbf{r},t)\hat{\psi}^\dagger(\mathbf{r}',t')|0\rangle &= \langle 0|\hat{U}^\dagger\hat{\psi}(\mathbf{r},t)\hat{\psi}^\dagger(\mathbf{r}',t')\hat{U}|0\rangle \\ &= \langle 0|\hat{\psi}(-\mathbf{r},t)\hat{\psi}^\dagger(-\mathbf{r}',t')|0\rangle,\end{aligned} \tag{2.31}$$

[5] Here we assume that the current operator has the same expression as in a free particle system. When the interaction part of the Hamiltonian is not invariant under a local U(1) transformation (see Sec. 2.3), there is an additional contribution due to the interactions.

where $|0\rangle$ denotes the ground state. A similar result holds for the time-ordered correlation function (i.e. the one-particle Green function), both at zero and finite temperatures,

$$G(\mathbf{r},t;\mathbf{r}',t') = G(-\mathbf{r},t;-\mathbf{r}',t'). \tag{2.32}$$

Similarly, for two operators $\hat{A}$ and $\hat{B}$ with signatures ϵ_A^P and ϵ_B^P, one has

$$\begin{aligned}\chi_{AB}(\mathbf{r},t;\mathbf{r}',t') &= \langle T\hat{A}(\mathbf{r},t)\hat{B}(\mathbf{r}',t')\rangle \\ &= \chi_{AB}(-\mathbf{r},t;-\mathbf{r}',t') = \epsilon_A^P\epsilon_B^P\chi_{AB}(\mathbf{r},t;\mathbf{r}',t')\end{aligned} \tag{2.33}$$

(and the analogous result in imaginary time). χ_{AB} vanishes if $\epsilon_A^P \neq \epsilon_B^P$. Equations (2.32) and (2.33) are also valid in the Matsubara formalism.

2.2.2 Space translation

As a second example, we consider the continuous transformations $\mathcal{T}(\mathbf{a})$ where the observer $\mathcal{O}'$ uses a reference frame translated by $-\mathbf{a}$ with respect to that used by $\mathcal{O}$. The set of transformations $\mathcal{T}(\mathbf{a})$ form an Abelian group, $\mathcal{T}(\mathbf{a})\mathcal{T}(\mathbf{b}) = \mathcal{T}(\mathbf{b})\mathcal{T}(\mathbf{a}) = \mathcal{T}(\mathbf{a}+\mathbf{b})$.

By analogy with classical mechanics, we define the Wigner operator $\hat{U}(\mathbf{a}) \equiv \hat{U}(\mathcal{T}(\mathbf{a}))$ for a single particle by

$$\begin{aligned}\hat{U}^\dagger(\mathbf{a})\hat{\mathbf{r}}\hat{U}(\mathbf{a}) &= \mathcal{U}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} + \mathbf{a}, \\ \hat{U}^\dagger(\mathbf{a})\hat{\mathbf{p}}\hat{U}(\mathbf{a}) &= \hat{\mathbf{p}}.\end{aligned} \tag{2.34}$$

The spin operator is invariant and is ignored in the following. Equations (2.34) define $\hat{U}$ only up to a phase. This ambiguity is lifted by specifying the action of $\hat{U}$ of the vectors $|\mathbf{r}\rangle$. A possible choice, compatible both with $\hat{U}^\dagger(\mathbf{a})\hat{\mathbf{p}}\hat{U}(\mathbf{a}) = \hat{\mathbf{p}}$ and the group structure of the translation transformations $\mathcal{T}(\mathbf{a})$, is

$$\hat{U}(\mathbf{a})|\mathbf{r}\rangle = |\mathbf{r}+\mathbf{a}\rangle. \tag{2.35}$$

The generator of the group is the momentum operator $\mathbf{p}$: Writing $\hat{U}(d\mathbf{a}) = 1 - id\mathbf{a}\cdot\hat{\mathbf{p}}$ (note the minus sign) for an infinitesimal transformation, and using $[\hat{r}_\mu, \hat{p}_{\mu'}] = i\delta_{\mu,\mu'}$ $(\mu,\mu' = x,y,\cdots)$, we verify that (2.34) are satisfied. Proceeding as in section 2.1.3 [see Eqs. (2.11)], we then find

$$\hat{U}(\mathbf{a}) = \exp(-i\mathbf{a}\cdot\hat{\mathbf{p}}). \tag{2.36}$$

The wavefunction transforms as $\Psi'(\mathbf{r}) = \Psi(\mathbf{r})$.

Many-particle system

The preceding results are readily extended to many-particle systems, where $\hat{U}(\mathbf{a}) = \exp(-i\mathbf{a}\cdot\hat{\mathbf{P}})$ with $\hat{\mathbf{P}}$ the total momentum operator. In second-quantized form, $\hat{\mathbf{P}} = \int d^dr\,\hat{\psi}^\dagger(\mathbf{r})(-i\boldsymbol{\nabla})\hat{\psi}(\mathbf{r})$. Translation invariance implies that the total momentum is conserved: $[\hat{\mathbf{P}},\hat{H}] = 0$ and $d\hat{\mathbf{P}}_H/dt = 0$.

Equation (2.28) with $\mathcal{U}(\mathbf{r}) = \mathbf{r}+\mathbf{a}$ shows that $\hat{\psi}^{(\dagger)}(\mathbf{r})$ transforms as

$$\hat{U}^\dagger(\mathbf{a})\hat{\psi}^{(\dagger)}(\mathbf{r})\hat{U}(\mathbf{a}) = \hat{\psi}^{(\dagger)}(\mathbf{r}-\mathbf{a}). \tag{2.37}$$

Any local operator $\hat{A}(\mathbf{r})$ therefore transforms as $\hat{U}^\dagger(\mathbf{a})\hat{A}(\mathbf{r})\hat{U}(\mathbf{a}) = \hat{A}(\mathbf{r}-\mathbf{a})$. Thus, in a translation-invariant system, correlation functions satisfy $\chi_{AB}(\mathbf{r},t;\mathbf{r}',t') = \chi_{AB}(\mathbf{r}-\mathbf{a},t;\mathbf{r}'-\mathbf{a},t')$.

2.2.3 Space rotation

A space rotation is defined by a 3×3 orthogonal matrix $\mathcal{U} \in SO(3)$. We first consider a single spinless particle. The Wigner operator corresponding to an infinitesimal rotation of angle $d\theta$ about an axis $\mathbf{n}$ is defined by

$$\begin{aligned} \hat{U}^\dagger(d\theta, \mathbf{n})\hat{\mathbf{r}}\hat{U}(d\theta, \mathbf{n}) &= \hat{\mathbf{r}} + d\theta\mathbf{n} \times \hat{\mathbf{r}}, \\ \hat{U}^\dagger(d\theta, \mathbf{n})\hat{\mathbf{p}}\hat{U}(d\theta, \mathbf{n}) &= \hat{\mathbf{p}} + d\theta\mathbf{n} \times \hat{\mathbf{p}}. \end{aligned} \tag{2.38}$$

The generator of the group is the orbital angular momentum $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$, i.e. $\hat{U}(d\theta, \mathbf{n}) = 1 - id\theta\mathbf{n} \cdot \hat{\mathbf{L}}$, as can be verified by direct substitution in (2.38) using the commutation relations $\hat{\mathbf{L}} \times \hat{\mathbf{L}} = i\hat{\mathbf{L}}$. Proceeding as in section 2.1.3, we obtain

$$\hat{U}(\theta, \mathbf{n}) = \exp(-i\theta\mathbf{n} \cdot \hat{\mathbf{L}}). \tag{2.39}$$

By choosing the arbitrary phase factor in (2.39) equal to unity, we ensure that the operators $\hat{U}(\theta, \mathbf{n})$ constitutes a representation of the group of rotations in ordinary space.

For a particle with spin, the Wigner operator can be written as

$$\hat{U}(\theta, \mathbf{n}) = \hat{U}_L(\theta, \mathbf{n})\hat{U}_S(\theta, \mathbf{n}), \tag{2.40}$$

where $\hat{U}_L(\theta, \mathbf{n})$ is given by (2.39) and $\hat{U}_S(\theta, \mathbf{n})$ acts in spin space. We determine $\hat{U}_S$ by requiring that the spin operator $\hat{\mathbf{S}}$ transforms as the orbital angular momentum operator $\hat{\mathbf{L}}$. Since $\hat{U}_L^\dagger(d\theta, \mathbf{n})\hat{\mathbf{L}}\hat{U}_L(d\theta, \mathbf{n}) = \hat{\mathbf{L}} + d\theta\mathbf{n} \times \hat{\mathbf{L}}$, we demand that

$$\hat{U}_S^\dagger(d\theta, \mathbf{n})\hat{\mathbf{S}}\hat{U}_S(d\theta, \mathbf{n}) = \hat{\mathbf{S}} + d\theta\mathbf{n} \times \hat{\mathbf{S}}. \tag{2.41}$$

Equation (2.41) gives $\hat{U}_S(d\theta, \mathbf{n}) = 1 - id\theta\mathbf{n} \cdot \hat{\mathbf{S}}$, as can be verified using the spin commutation relations $\hat{\mathbf{S}} \times \hat{\mathbf{S}} = i\hat{\mathbf{S}}$, and

$$\hat{U}_S(\theta, \mathbf{n}) = \exp(-i\theta\mathbf{n} \cdot \hat{\mathbf{S}}). \tag{2.42}$$

We thus obtain

$$\hat{U}(\theta, \mathbf{n}) = \hat{U}_L(\theta, \mathbf{n})\hat{U}_S(\theta, \mathbf{n}) = \exp(-i\theta\mathbf{n} \cdot \hat{\mathbf{J}}), \tag{2.43}$$

where $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ is the total angular momentum operator.

The wave function $\Psi_\sigma(\mathbf{r})$ transforms into

$$\begin{aligned} \Psi'(\mathbf{r}) &= \langle \mathbf{r}, \sigma | \hat{U}(\theta, \mathbf{n}) | \Psi \rangle \\ &= \sum_{\sigma'} \int d^d r' \langle \mathbf{r}, \sigma | \hat{U}(\theta, \mathbf{n}) | \mathbf{r}', \sigma' \rangle \langle \mathbf{r}', \sigma' | \Psi \rangle \\ &= \sum_{\sigma'} \big(\hat{U}_S(\theta, \mathbf{n})\big)_{\sigma,\sigma'} \Psi_{\sigma'}(\mathcal{U}^{-1}(\mathbf{r})), \end{aligned} \tag{2.44}$$

where $\mathcal{U}(\mathbf{r})$ is the image of $\mathbf{r}$ by a rotation of angle θ about the $\mathbf{n}$ axis. We have introduced the eigenstates $|\sigma\rangle$ $(\sigma = -S, \cdots, S)$ of the operator $\hat{S}^z$, where S is the spin of the particle.

Many-particle system

The Wigner operator is given by (2.43) where $\hat{\mathbf{J}}$ should now be understood as the total angular momentum operator of the many-particle system. The annihilation operator $\hat\psi(\mathbf{r})$ transforms as the wave function [Eq. (2.44)], i.e.

$$\hat U^\dagger(\theta,\mathbf{n})\hat\psi^{(\dagger)}_\sigma(\mathbf{r})\hat U(\theta,\mathbf{n}) = \sum_{\sigma'}\big(\hat U_S(\theta,\mathbf{n})\big)_{\sigma,\sigma'}\hat\psi^{(\dagger)}_{\sigma'}(\mathcal{U}^{-1}(\mathbf{r})). \tag{2.45}$$

Rotation invariance implies the conservation of the total angular momentum $\hat{\mathbf{J}}$.[6]

In a rotation-invariant system, the one-particle Green function satisfies

$$G_{\sigma\sigma'}(\mathbf{r},t;\mathbf{r}',t') = \sum_{\sigma_1,\sigma'_1}\big(\hat U_S(\theta,\mathbf{n})\big)_{\sigma,\sigma_1}\big(\hat U_S(\theta,\mathbf{n})\big)_{\sigma',\sigma'_1}G_{\sigma_1\sigma'_1}(\mathcal{U}^{-1}(\mathbf{r}),t;\mathcal{U}^{-1}(\mathbf{r}'),t'). \tag{2.46}$$

For two scalar operators $\hat A$ and $\hat B$, rotation invariance implies $\chi_{AB}(\mathbf{r},t;\mathbf{r}',t') = \chi_{AB}(\mathcal{U}^{-1}(\mathbf{r}),t;\mathcal{U}^{-1}(\mathbf{r}'),t')$.

From the preceding results, one can easily deduce the Hamiltonian in a reference frame rotating about an arbitrary axis $\mathbf{n}$ with angular velocity $\boldsymbol\omega = \omega\mathbf{n}$. The Wigner operator $\hat U(\theta(t),\mathbf{n})$ is given by (2.43) with $\theta(t) = -\omega t$. The Hamiltonian in the rotating frame is given by

$$\begin{aligned}\hat H' &= \hat U(\theta(t),\mathbf{n})\hat H\hat U^\dagger(\theta(t),\mathbf{n}) - \boldsymbol\omega\cdot\hat{\mathbf{J}} \\ &= \hat H - \boldsymbol\omega\cdot\hat{\mathbf{J}}\end{aligned} \tag{2.47}$$

(see Eq. (2.14)), where $\hat{\mathbf{J}}$ should be understood as the angular momentum operator in the new frame. The last result in (2.47) holds for a rotation-invariant system ($[\hat H,\hat{\mathbf{J}}]=0$).

2.2.4 Time translation

When the clocks used by the two observers are shifted with respect to one another, i.e. $t(\mathcal{O}') = t(\mathcal{O}) - \tau$, the Wigner operator $\hat U_\tau(t)$ is the evolution operator $\hat T(t+\tau,t)$ (up to a phase factor that we take equal to unity),

$$|\Psi'(t)\rangle = |\Psi(t+\tau)\rangle = \hat T(t+\tau,t)|\Psi(t)\rangle. \tag{2.48}$$

The Wigner operators satisfy the group property $\hat T(t_1,t_2)\hat T(t_2,t_3) = \hat T(t_1,t_3)$. For an infinitesimal time translation, $|\Psi(t+d\tau)\rangle = |\Psi(t)\rangle + d\tau\partial_t|\Psi(t)\rangle = (1-i\hat H(t)d\tau)|\Psi(t)\rangle$, so that

$$\hat U_{d\tau}(t) = 1 - i\hat H(t)d\tau. \tag{2.49}$$

The Hamiltonian is the generator of time translation. Given that

$$i\partial_t|\Psi'(t)\rangle = i\partial_t|\Psi(t+\tau)\rangle = \hat H(t+\tau)|\Psi(t+\tau)\rangle = \hat H(t+\tau)|\Psi'(t)\rangle, \tag{2.50}$$

we obtain

$$\hat H'(t) = \hat H(t+\tau). \tag{2.51}$$

Thus, time translation invariance implies that the Hamiltonian $\hat H'(t) = \hat H(t+\tau) = \hat H(t)$ is time independent (the same conclusion can be reached by using (2.16)). In that case the evolution operator is simply $\hat U_\tau(t) = \hat T(t+\tau,t) = e^{-i\hat H\tau}$.

[6] If the system is invariant in a rotation in orbital and spin spaces, separately, then both $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ are conserved.

2.2.5 Galilean transformation

Let us consider the case where the observer $\mathcal{O}'$ uses a reference frame which moves with velocity $\mathbf{v}$ with respect to that used by $\mathcal{O}$. The corresponding Wigner operator is defined by

$$\begin{aligned} \hat{U}^\dagger(\mathbf{v},t)\hat{\mathbf{r}}\hat{U}(\mathbf{v},t) &= \hat{\mathbf{r}} - \mathbf{v}t, \\ \hat{U}^\dagger(\mathbf{v},t)\hat{\mathbf{p}}\hat{U}(\mathbf{v},t) &= \hat{\mathbf{p}} - m\mathbf{v}, \end{aligned} \tag{2.52}$$

where m is the mass of the particle (we ignore the spin of the particle which is supposed to remain unaltered). We assume that $\hat{U}(\mathbf{v},t)$ can be written as the product $\hat{U}_p(\mathbf{v},t)\hat{U}_r(\mathbf{v},t)$ of two unitary operators where $\hat{U}_p$ ($\hat{U}_r$) is a function of $\hat{\mathbf{p}}$ ($\hat{\mathbf{r}}$) only. If we impose

$$\begin{aligned} \hat{U}_p^\dagger(\mathbf{v},t)\hat{\mathbf{r}}\hat{U}_p(\mathbf{v},t) &= \hat{\mathbf{r}} - \mathbf{v}t, \\ \hat{U}_r^\dagger(\mathbf{v},t)\hat{\mathbf{p}}\hat{U}_r(\mathbf{v},t) &= \hat{\mathbf{p}} - m\mathbf{v}, \end{aligned} \tag{2.53}$$

equations (2.52) are automatically satisfied. $\hat{U}_p(\mathbf{v},t)$ is a translation operator and can be written as $\hat{U}_p(\mathbf{v},t) = \exp(i\gamma_p(\mathbf{v},t) + it\mathbf{v}\cdot\hat{\mathbf{p}})$ with $\gamma_p(\mathbf{v},t)$ an arbitrary phase (see Sec. 2.2.2). $\hat{U}_r(\mathbf{v},t)$ translates $\hat{\mathbf{p}}$ but leaves $\hat{\mathbf{r}}$ invariant. Following the analysis of section 2.2.2, we find $\hat{U}_r(\mathbf{v},t) = \exp(i\gamma_r(\mathbf{v},t) - im\mathbf{v}\cdot\hat{\mathbf{r}})$ with $\gamma_r(\mathbf{v},t)$ an arbitrary phase. This gives

$$\begin{aligned} \hat{U}(\mathbf{v},t) &= \exp\bigl(i\gamma_p(\mathbf{v},t) + i\gamma_r(\mathbf{v},t)\bigr)\exp(it\mathbf{v}\cdot\hat{\mathbf{p}})\exp(-im\mathbf{v}\cdot\hat{\mathbf{r}}) \\ &= \exp\Bigl[i\Bigl(\gamma_p(\mathbf{v},t) + \gamma_r(\mathbf{v},t) - \frac{1}{2}m\mathbf{v}^2 t\Bigr) + i(t\hat{\mathbf{p}} - m\hat{\mathbf{r}})\cdot\mathbf{v}\Bigr], \end{aligned} \tag{2.54}$$

where we have used the operator identity $e^{\hat{A}}e^{\hat{B}} = e^{\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{A}+\hat{B}}$ for two operators which commutes with their commutator. If we choose $\gamma_p(\mathbf{v},t) + \gamma_r(\mathbf{v},t) = \frac{1}{2}m\mathbf{v}^2 t$, then the operators

$$\hat{U}(\mathbf{v},t) = \exp[i(\hat{\mathbf{p}}t - m\hat{\mathbf{r}})\cdot\mathbf{v}] \tag{2.55}$$

form a group with the composition law $\hat{U}(\mathbf{v}_1)\hat{U}(\mathbf{v}_2) = \hat{U}(\mathbf{v}_1 + \mathbf{v}_2)$. The generator of the group is $-\hat{\mathbf{p}}t + m\hat{\mathbf{r}}$. It turns out that the choice $\gamma(\mathbf{v},t) = \gamma_p(\mathbf{v},t) + \gamma_r(\mathbf{v},t) = \frac{1}{2}m\mathbf{v}^2 t + \gamma_0(\mathbf{v})$, with $\gamma_0(\mathbf{v})$ a time-independent constant, is the only one compatible with invariance under a Galilean transformation. Indeed, for a free particle with Hamiltonian $\hat{H} = \hat{\mathbf{p}}^2/2m$, Galilean invariance [Eq. (2.16)] implies $d\gamma(\mathbf{v},t)/dt = \frac{1}{2}m\mathbf{v}^2$, i.e. $\gamma(\mathbf{v},t) = \frac{1}{2}m\mathbf{v}^2 t + \gamma_0(\mathbf{v})$.

The wave function $\Psi(\mathbf{r},t)$ transforms as

$$\begin{aligned} \Psi'(\mathbf{r}',t) &\equiv \langle\mathbf{r}'|\hat{U}(\mathbf{v},t)|\Psi(t)\rangle \\ &= \int d^d r'' \sum_{\mathbf{p}} e^{i\gamma(\mathbf{v},t)} \langle\mathbf{r}'|\exp(it\mathbf{v}\cdot\hat{\mathbf{p}})|\mathbf{p}\rangle\langle\mathbf{p}|\exp(-im\mathbf{v}\cdot\hat{\mathbf{r}})|\mathbf{r}''\rangle\langle\mathbf{r}''|\Psi(t)\rangle \\ &= \int d^d r'' \int \frac{d^d p}{(2\pi)^d} e^{i\gamma(\mathbf{v},t) - im\mathbf{v}\cdot\mathbf{r}'' + i\mathbf{p}\cdot(\mathbf{v}t + \mathbf{r}' - \mathbf{r}'')}\Psi(\mathbf{r}'',t) \\ &= \int d^d r''\, e^{i\gamma(\mathbf{v},t) - im\mathbf{v}\cdot\mathbf{r}''}\delta(\mathbf{v}t + \mathbf{r}' - \mathbf{r}'')\Psi(\mathbf{r}'',t) \\ &= e^{\frac{i}{2}m\mathbf{v}^2 t - im\mathbf{v}\cdot\mathbf{r}}\Psi(\mathbf{r},t), \end{aligned} \tag{2.56}$$

where $\mathbf{r} = \mathbf{r}' + \mathbf{v}t$.

Many-particle system

For a many-particle system, the Wigner operator reads

$$\hat{U}(\mathbf{v},t) = \exp[i(\hat{\mathbf{P}}t - M\hat{\mathbf{R}})\cdot\mathbf{v}], \tag{2.57}$$

where $M = Nm$ is the total mass of the N particles, $\hat{\mathbf{R}}$ and $\hat{\mathbf{P}}$ the position and momentum operators of the center of mass. In second quantized form, $M\hat{\mathbf{R}} = m\int d^dr\,\hat{\psi}^\dagger(\mathbf{r})\hat{\mathbf{r}}\hat{\psi}(\mathbf{r})$. The expression of $\hat{\mathbf{P}}$ is given is section 2.2.2. Galilean invariance implies

$$\frac{d[\hat{\mathbf{P}}t - M\hat{\mathbf{R}}]_H}{dt} = 0. \tag{2.58}$$

If the system is translation invariant, this gives $\hat{\mathbf{P}}_H = Md\hat{\mathbf{R}}_H/dt$.

It is easy to verify that a system of N interacting particles with Hamiltonian $\hat{H} = \sum_i \hat{\mathbf{p}}_i^2/2m + \hat{V}$, where $\hat{V} \equiv \hat{V}(\{\mathbf{r}_i - \mathbf{r}_j\})$ depends only the relative particles' positions, is Galilean invariant. Indeed one finds[7]

$$\begin{aligned}\hat{H}' &= \hat{U}(\mathbf{v},t)\hat{H}\hat{U}^\dagger(\mathbf{v},t) + i\big(\partial_t\hat{U}(\mathbf{v},t)\big)\hat{U}^\dagger(\mathbf{v},t) \\ &= \sum_i \frac{(\hat{\mathbf{p}}_i + m\mathbf{v})^2}{2m} + \hat{V} - \frac{M}{2}\mathbf{v}^2 - \mathbf{v}\cdot\hat{\mathbf{P}} = \hat{H},\end{aligned} \tag{2.59}$$

where $\hat{\mathbf{P}} = \sum_i \hat{\mathbf{p}}_i$. The mean value of the energy is however not the same for $\mathcal{O}$ and $\mathcal{O}'$, since

$$\langle \hat{H}'\rangle_{\mathcal{O}'} = \langle \hat{H}\rangle_{\mathcal{O}} - \mathbf{v}\cdot\langle\hat{\mathbf{P}}\rangle_{\mathcal{O}} + \frac{1}{2}Nm\mathbf{v}^2 \tag{2.60}$$

using (2.15). Equation (2.60) is the relation obtained in classical mechanics.

Since the operator $\hat{\psi}(\mathbf{r})$ transforms as the wavefunction, we have

$$\begin{aligned}\hat{\psi}'(\mathbf{r}') &= \hat{U}^\dagger(\mathbf{v},t)\hat{\psi}(\mathbf{r}')\hat{U}(\mathbf{v},t) = e^{\frac{i}{2}m\mathbf{v}^2t - im\mathbf{v}\cdot\mathbf{r}}\hat{\psi}(\mathbf{r}), \\ \hat{\psi}'^\dagger(\mathbf{r}') &= \hat{U}^\dagger(\mathbf{v},t)\hat{\psi}^\dagger(\mathbf{r}')\hat{U}(\mathbf{v},t) = e^{-\frac{i}{2}m\mathbf{v}^2t + im\mathbf{v}\cdot\mathbf{r}}\hat{\psi}^\dagger(\mathbf{r})\end{aligned} \tag{2.61}$$

(note that the Schrödinger representation operator $\hat{\psi}'(\mathbf{r}') \equiv \hat{\psi}'(\mathbf{r}',t)$ is actually time dependent). From (2.61), we obtain for the density and current operators,

$$\begin{aligned}\hat{n}'(\mathbf{r}') &= \hat{U}^\dagger(\mathbf{v},t)\hat{n}(\mathbf{r}')\hat{U}(\mathbf{v},t) = \hat{n}(\mathbf{r}), \\ \hat{\mathbf{j}}'(\mathbf{r}') &= \hat{U}^\dagger(\mathbf{v},t)\hat{\mathbf{j}}(\mathbf{r}')\hat{U}(\mathbf{v},t) = \hat{\mathbf{j}}(\mathbf{r}) - \hat{n}(\mathbf{r})\mathbf{v}.\end{aligned} \tag{2.62}$$

The transformation of correlation functions is easily deduced from (2.61).

2.2.6 Time reversal

Antiunitary operators

Before discussing time reversal symmetry, we should recall the definition and properties of antilinear and antiunitary operators. An operator $\hat{A}$ is antilinear if

$$\hat{A}(\alpha|\Psi\rangle + \beta|\Phi\rangle) = \alpha^*\hat{A}|\Psi\rangle + \beta^*\hat{A}|\Phi\rangle, \tag{2.63}$$

[7] We use the first expression of $\hat{U}(\mathbf{v},t)$ in (2.54), generalized to the case of N particles.

where α and β are complex numbers. Antilinearity implies that $[\langle\Psi|(\hat{A}|\Phi\rangle)]^*$ is a linear function of $|\Phi\rangle$, i.e. $[\langle\Psi|(\hat{A}(\alpha_1|\Phi_1\rangle + \alpha_2|\Phi_2\rangle))]^* = \alpha_1[\langle\Psi|(\hat{A}|\Phi_1\rangle)]^* + \alpha_2[\langle\Psi|(\hat{A}|\Phi_2\rangle)]^*$. This defines a bra that we denote by $\langle\Psi|\hat{A}$:

$$(\langle\Psi|\hat{A})|\Phi\rangle = [\langle\Psi|(\hat{A}|\Phi\rangle)]^*. \tag{2.64}$$

The Dirac notation should be used with care with antilinear operators since $(\langle\Psi|\hat{A})|\Phi\rangle \neq \langle\Psi|(\hat{A}|\Phi\rangle)$; one should always specify whether the operator acts on the ket or the bra. The adjoint $\hat{A}^\dagger$ is defined by $(|\Psi\rangle, \hat{A}|\Phi\rangle) = (\hat{A}^\dagger|\Psi\rangle, |\Phi\rangle)^*$, where $(\cdot,\cdot)$ denotes the scalar product.[8] In Dirac's notation,

$$\langle\Psi|(\hat{A}|\Phi\rangle) = \langle\Phi|(\hat{A}^\dagger|\Psi\rangle). \tag{2.65}$$

If each of the operators $\hat{A}_1$ and $\hat{A}_2$ is either linear or antilinear, $(\hat{A}_1\hat{A}_2)^\dagger = \hat{A}_2^\dagger\hat{A}_1^\dagger$. A product of N_1 linear and N_2 antilinear operators is linear (antilinear) if N_2 is even (odd).

An antilinear operator $\hat{A}$ is said to be antiunitary if $\hat{A}^\dagger\hat{A} = 1$. One then has

$$(\hat{A}|\Psi\rangle, \hat{A}|\Phi\rangle) = (\langle\Psi|\hat{A}^\dagger)(\hat{A}|\Phi\rangle) = \langle\Psi|\Phi\rangle^* = (|\Psi\rangle, |\Phi\rangle)^*. \tag{2.66}$$

This equation shows that an antiunitary operator conserves the modulus $|\langle\Psi|\Phi\rangle|$ of the scalar product and is therefore a possible candidate as a Wigner operator associated to a symmetry transformation.

Wigner operator for a spin-zero particle

After these technical preliminaries, we can turn back to our two observers $\mathcal{O}$ and $\mathcal{O}'$ and discuss time reversal symmetry. In this section, we consider only time independent Hamiltonians. We assume that the two observers use clocks moving opposite in time with respect to one another, i.e.

$$t(\mathcal{O}') = -t(\mathcal{O}) \tag{2.67}$$

if we take the two clocks to coincide at $t = 0$.

Let us start with classical mechanics. If $\mathcal{O}$ describes a classical particle by its position $\mathbf{r}(t)$ and momentum $\mathbf{p}(t)$, $\mathcal{O}'$ describes the same particle by $\mathbf{r}'(t) = \mathbf{r}(-t)$ and $\mathbf{p}'(t) = -\mathbf{p}(-t)$. Time reversal invariance means that if $(\mathbf{r}(t), \mathbf{p}(t))$ is a possible trajectory, then $(\mathbf{r}'(t), \mathbf{p}'(t))$ is also a solution of the equations of motion $\dot{\mathbf{r}} = \partial H/\partial p$ and $\dot{\mathbf{p}} = -\partial H/\partial\mathbf{r}$. Clearly, this requires the Hamiltonian to be an even function of $\mathbf{p}$: $H(\mathbf{p}, \mathbf{r}) = H(-\mathbf{p}, \mathbf{r})$. The quantum Hamiltonian $H(-i\boldsymbol{\nabla}, \mathbf{r})$ is then real in the coordinate representation.[9]

In quantum mechanics, time reversal symmetry defines a Wigner operator $\hat{U}(t)$ by

$$|\Psi'(t)\rangle = \hat{U}(t)|\Psi(t)\rangle, \tag{2.68}$$

where $|\Psi(t)\rangle$ and $|\Psi'(t)\rangle$ are the vectors assigned to the system by $\mathcal{O}$ and $\mathcal{O}'$ at the same subjective time t. It is convenient to introduce the operator $\hat{\Theta}$ relating the vectors $|\Psi\rangle$ and $|\Psi'\rangle$ at the same objective time (t for $\mathcal{O}'$ and $-t$ for $\mathcal{O}$),

$$|\Psi'(t)\rangle = \hat{\Theta}|\Psi(-t)\rangle. \tag{2.69}$$

[8] The usual definition of the adjoint of a linear operator, $(|\Psi\rangle, \hat{A}|\Phi\rangle) = (\hat{A}^\dagger|\Psi\rangle, |\Phi\rangle)$, cannot be used since the lhs is here linear in $|\Psi\rangle$ while the rhs would be antilinear if $\hat{A}$ were antilinear.

[9] The Hamiltonian $H(-i\boldsymbol{\nabla}, \mathbf{r}) = -\boldsymbol{\nabla}^2/2m + V(\mathbf{r})$ of a particle moving in a potential $V(\mathbf{r})$ is real. When the particle is subjected to a vector potential, the Hamiltonian is not real any more and time-reversal invariance is broken.

We shall see that $\hat{\Theta}$ is time independent. $\hat{U}(t)$ and $\hat{\Theta}$ are related by $\hat{U}(t) = \hat{\Theta}e^{2i\hat{H}t} = e^{-2i\hat{H}'t}\hat{\Theta}$ where $\hat{H}'$ is the Hamiltonian used by $\mathcal{O}'$.[10] According to Wigner's theorem, $\hat{U}(t)$ and $\hat{\Theta}$ are both either unitary or antiunitary. In both cases, they satisfy $\hat{U}^\dagger(t)\hat{U}(t) = \hat{\Theta}^\dagger\hat{\Theta} = 1$.

By analogy with classical mechanics, we require

$$\begin{aligned} \langle\Psi'(t)|\hat{\mathbf{r}}|\Psi'(t)\rangle &= \langle\Psi(-t)|\hat{\mathbf{r}}|\Psi(-t)\rangle, \\ \langle\Psi'(t)|\hat{\mathbf{p}}|\Psi'(t)\rangle &= -\langle\Psi(-t)|\hat{\mathbf{p}}|\Psi(-t)\rangle. \end{aligned} \tag{2.70}$$

This implies

$$\begin{aligned} \hat{\Theta}^\dagger\hat{\mathbf{r}}\hat{\Theta} &= \hat{\mathbf{r}}, \\ \hat{\Theta}^\dagger\hat{\mathbf{p}}\hat{\Theta} &= -\hat{\mathbf{p}}. \end{aligned} \tag{2.71}$$

The preceding equations define $\hat{\Theta}$ only up to a phase factor. We fix this phase factor by specifying the action of $\hat{\Theta}$ on the eigenvectors $|\mathbf{r}\rangle$ of the position operator,

$$\hat{\Theta}|\mathbf{r}\rangle = |\mathbf{r}\rangle. \tag{2.72}$$

It is easy to show that this equation implies $\hat{\Theta}|\mathbf{p}\rangle = |\mathbf{p}\rangle$ if $\hat{\Theta}$ is linear and $\hat{\Theta}|\mathbf{p}\rangle = |-\mathbf{p}\rangle$ if $\hat{\Theta}$ is antilinear.[11] Since equations (2.71) imply $\hat{\Theta}|\mathbf{p}\rangle = |-\mathbf{p}\rangle$, we conclude that $\hat{\Theta}$ must be antilinear (and therefore antiunitary). The antilinearity of $\hat{\Theta}$ can also be seen by evaluating $\hat{\Theta}^\dagger i\hat{\Theta} = \hat{\Theta}^\dagger[\hat{r}_\mu, \hat{p}_\mu]\hat{\Theta} = -[\hat{r}_\mu, \hat{p}_\mu] = -i$.

The time reversed Hamiltonian $\hat{H}'$ is deduced from the equation of motion of the time reversed vector $|\Psi'(t)\rangle$,

$$\begin{aligned} i\partial_t|\Psi'(t)\rangle = i\partial_t\hat{\Theta}|\Psi(-t)\rangle = -\hat{\Theta}i\partial_t|\Psi(-t)\rangle = \hat{\Theta}\hat{H}|\Psi(-t)\rangle = \hat{\Theta}\hat{H}\hat{\Theta}^\dagger|\Psi'(t)\rangle \\ \equiv \hat{H}'|\Psi'(t)\rangle, \end{aligned} \tag{2.73}$$

i.e.

$$\hat{H}' = \hat{\Theta}\hat{H}\hat{\Theta}^\dagger. \tag{2.74}$$

Time reversal invariance means that $|\Psi'(t)\rangle$ and $|\Psi(t)\rangle$ satisfy the same Schrödinger equation, $\hat{H}' = \hat{H}$, i.e.

$$[\hat{H}, \hat{\Theta}] = 0. \tag{2.75}$$

If $\hat{\Theta}$ were linear, we would obtain $\hat{H}' = -\hat{\Theta}\hat{H}\hat{\Theta}^\dagger$ and therefore $\hat{H}\hat{\Theta} + \hat{\Theta}\hat{H} = 0$ in a time-reversal invariant system. From the latter equation, we deduce that if $|\Psi(t=0)\rangle$ is an eigenstate with energy E, then the time reversed vector $\hat{\Theta}|\Psi(t=0)\rangle$ is an eigenstate with energy $-E$: $\hat{H}\hat{\Theta}|\Psi\rangle = -\hat{\Theta}\hat{H}|\Psi\rangle = -E\hat{\Theta}|\Psi\rangle$. The elementary case of a free particle is sufficient to show that this result is meaningless, since the momentum eigenstate $|\mathbf{p}\rangle$ and $\hat{\Theta}|\mathbf{p}\rangle = |-\mathbf{p}\rangle$ are eigenstates of the Hamiltonian with the same energy $\mathbf{p}^2/2m$. Note that time reversal invariance [Eq. (2.75)] does not imply a conservation law as for the other symmetry transformations. Since $\hat{\Theta}$ is antilinear, its equation of motion in the Heisenberg picture, $i\partial_t\hat{\Theta}_H = -\hat{H}\hat{\Theta}_H - \hat{\Theta}_H\hat{H}$, involves the anticommutator of $\hat{H}$ and $\hat{\Theta}$ instead of the commutator.

[10] We use the fact that the time time-evolution operators are $\hat{T}(t+\tau,t) = e^{-i\hat{H}\tau}$ and $\hat{T}'(t+\tau,t) = e^{-i\hat{H}'\tau}$ for $\mathcal{O}$ and $\mathcal{O}'$, respectively.

[11] This follows from $\hat{\Theta}|\mathbf{p}\rangle = \hat{\Theta}\int d^dr|\mathbf{r}\rangle\langle\mathbf{r}|\mathbf{p}\rangle = \int d^dr\langle\mathbf{r}|\mathbf{p}\rangle^*\hat{\Theta}|\mathbf{r}\rangle = \int d^dr\langle\mathbf{r}|-\mathbf{p}\rangle|\mathbf{r}\rangle = |-\mathbf{p}\rangle$ when $\hat{\Theta}$ is antilinear.

In order to find an explicit expression of the operator $\hat\Theta$, we consider the wave function in the coordinate representation,

$$\begin{aligned}\Psi'(\mathbf{r},t) = \langle \mathbf{r}|\Psi'(t)\rangle = \langle \mathbf{r}|(\hat\Theta|\Psi(-t)\rangle) = \int d^d r' \langle \mathbf{r}|(\hat\Theta|\mathbf{r}'\rangle\langle \mathbf{r}'|\Psi(-t)\rangle) \\ = \int d^d r'\, \Psi^*(\mathbf{r}',-t)\langle \mathbf{r}|(\hat\Theta|\mathbf{r}'\rangle) = \Psi^*(\mathbf{r},-t). \end{aligned} \tag{2.76}$$

This shows that for a spin-zero particle, $\hat\Theta$ is the complex conjugation operator $\hat K$ in the coordinate representation,

$$\Psi'(\mathbf{r},t) = \hat K \Psi(\mathbf{r},-t) = \Psi^*(\mathbf{r},-t). \tag{2.77}$$

One can verify that $\Psi^*(\mathbf{r},-t)$ is a solution of the Schrödinger equation when the Hamiltonian is real in the coordinate representation. By taking the complex conjugate of $i\partial_t\Psi(\mathbf{r},t) = \hat H\Psi(\mathbf{r},t)$, one indeed obtains

$$i\partial_t \Psi^*(\mathbf{r},-t) = \hat H \Psi^*(\mathbf{r},-t). \tag{2.78}$$

Particle with spin

So far, we have considered only a spin-zero particle. Since the orbital angular momentum transforms as $\hat\Theta^\dagger \hat{\mathbf{r}}\times\hat{\mathbf{p}}\hat\Theta = -\hat{\mathbf{r}}\times\hat{\mathbf{p}}$, we demand that $\hat\Theta$ reverses the direction of the spin,

$$\hat\Theta^\dagger \hat{\mathbf{S}} \hat\Theta = -\hat{\mathbf{S}}. \tag{2.79}$$

Let us now denote by $\hat K$ the complex conjugation operator in the $(\mathbf{r}, S^z)$ representation. With the standard choice of the eigenstates $|\mathbf{r},\sigma\rangle$ ($\sigma = -S,\cdots,S$) of $\hat S^z$, the spin operators satisfy[12,13]

$$\begin{aligned} \hat K^\dagger \hat S^x \hat K &= (\hat S^x)^* = \hat S^x, \\ \hat K^\dagger \hat S^y \hat K &= (\hat S^y)^* = -\hat S^y, \\ \hat K^\dagger \hat S^x \hat K &= (\hat S^z)^* = \hat S^z. \end{aligned} \tag{2.80}$$

Thus $\hat K$ performs the time-reversal transformation for S^y but not for S^x and S^z. It is clear that $\hat\Theta$ can be written as the product $\hat Y \hat K$ of $\hat K$ and the unitary operator $\hat Y = \exp(-i\pi \hat S^y)$ that performs a rotation of angle π about the y axis. Thus we obtain

$$\hat\Theta = \eta e^{-i\pi \hat S^y} \hat K, \tag{2.81}$$

where η is an arbitrary complex number of modulus unity. Equation (2.81) yields[13] $\hat\Theta^2 = e^{-2i\pi\hat S^y} = (-1)^{2S}$, i.e. $\hat\Theta^2 = 1$ for a boson and $\hat\Theta^2 = -1$ for a fermion, regardless of the value of the phase factor η ($\eta = 1$ in the following). Combined with $\hat\Theta^\dagger\hat\Theta = 1$, this implies $\hat\Theta^\dagger = \hat\Theta$ (boson) and $\hat\Theta^\dagger = -\hat\Theta$ (fermion). For N fermions we would obtain $\hat\Theta^2 = (-1)^N$.[14]

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[12] This comes from $\langle \mathbf{r},\sigma|(\hat K^\dagger \hat S^\nu \hat K)|\mathbf{r}',\sigma'\rangle = \langle \mathbf{r},\sigma|(\hat K^\dagger \hat S^\nu|\mathbf{r}',\sigma'\rangle) = (\hat S^\nu_{\sigma,\sigma'})^*\langle \mathbf{r},\sigma|(\hat K^\dagger|\mathbf{r}',\sigma'\rangle) = (\hat S^\nu_{\sigma,\sigma'})^*\langle \mathbf{r}',\sigma'|\mathbf{r},\sigma\rangle = (\hat S^\nu_{\sigma,\sigma'})^*\langle \mathbf{r},\sigma|\mathbf{r}',\sigma'\rangle$ and the fact that $\hat S^\nu_{\sigma,\sigma'} = \langle\sigma|\hat S^\nu|\sigma'\rangle$ is real if $\nu = x,z$ and imaginary if $\nu = y$.

[13] Note that $\hat K^\dagger = \hat K$ and $\hat K^2 = 1$.

[14] This result is at the origin of Kramers' degeneracy in time-reversal invariant systems with an odd number of fermions (i.e. a half-integer total spin). In that case, any eigenstate $|\Psi\rangle$ of the Hamiltonian is degenerate with $\hat\Theta|\Psi\rangle$ whereas $\langle\Psi|(\hat\Theta|\Psi\rangle) = 0$, so that every energy level is at least doubly degenerate.

Many-particle systems

Equation (2.81) is readily generalized to a many-particle system; $\hat{\mathbf{S}}$ then denotes the total spin operator. For any operator $\hat{A}$, the matrix element between time-reversed states is given by

$$(\langle\Psi|\hat{\Theta}^\dagger)\hat{A}(\hat{\Theta}|\Phi\rangle) = \langle\Psi|(\hat{\Theta}^\dagger\hat{A}\hat{\Theta})|\Phi\rangle^* = \langle\Phi|(\hat{\Theta}^\dagger\hat{A}^\dagger\hat{\Theta})|\Psi\rangle. \tag{2.82}$$

For the operator $\hat{A}(t) = e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t}$ defined in the Heisenberg representation,

$$(\langle\Psi|\hat{\Theta}^\dagger)\hat{A}(t)(\hat{\Theta}|\Phi\rangle) = \langle\Phi|e^{-i\hat{H}t}(\hat{\Theta}^\dagger\hat{A}^\dagger\hat{\Theta})e^{i\hat{H}t}|\Psi\rangle = \epsilon_A^T\langle\Phi|\hat{A}^\dagger(-t)|\Psi\rangle, \tag{2.83}$$

when the system is time-reversal invariant ($[\hat{H},\hat{\Theta}]=0$). The last result in (2.83) holds if $\hat{A}$ is even or odd under time reversal: $\hat{\Theta}^\dagger\hat{A}\hat{\Theta} = \epsilon_A^T\hat{A}$ and $\hat{\Theta}^\dagger\hat{A}(t)\hat{\Theta} = \epsilon_A^T\hat{A}(-t)$ with $\epsilon_A^T = \pm$ the signature.[15] Let us now consider the zero-temperature correlation function $\chi_{AB}(t,t') = \langle 0|\hat{A}(t)\hat{B}(t')|0\rangle$, where $|0\rangle$ denotes the ground state. If the system is time-reversal invariant, $\hat{H}'=\hat{H}$, $\chi_{AB}(t,t')$ must satisfy[16]

$$\chi_{AB}(t,t') = (\langle 0|\hat{\Theta}^\dagger)\hat{A}(t)\hat{B}(t')(\hat{\Theta}|0\rangle). \tag{2.84}$$

Using (2.82), one finds

$$\begin{aligned}\chi_{AB}(t,t') &= \langle 0|(\hat{\Theta}^\dagger\hat{B}^\dagger(t')\hat{A}^\dagger(t)\hat{\Theta})|0\rangle \\ &= \langle 0|(e^{-i\hat{H}t'}\hat{\Theta}^\dagger\hat{B}^\dagger\hat{\Theta}e^{i\hat{H}t'}e^{-i\hat{H}t}\hat{\Theta}\hat{A}^\dagger\hat{\Theta}e^{i\hat{H}t})|0\rangle \\ &= \chi_{\Theta^\dagger\hat{B}^\dagger\hat{\Theta},\Theta^\dagger\hat{A}^\dagger\Theta}(-t',-t),\end{aligned} \tag{2.85}$$

i.e.

$$\chi_{AB}(\omega) = \chi_{\Theta^\dagger\hat{B}^\dagger\hat{\Theta},\Theta^\dagger\hat{A}^\dagger\Theta}(\omega) = \epsilon_A^T\epsilon_B^T\chi_{B^\dagger A^\dagger}(\omega) \tag{2.86}$$

where the last result holds for two operators with signatures ϵ_A^T and ϵ_B^T. Equations (2.82) and (2.85) also hold for the imaginary-time correlation function,[17]

$$\begin{aligned}\chi_{AB}(\tau,\tau') &= \langle 0|\hat{A}(\tau)\hat{B}(\tau')|0\rangle = \chi_{\Theta^\dagger\hat{B}^\dagger\hat{\Theta},\Theta^\dagger\hat{A}^\dagger\Theta}(-\tau',-\tau), \\ \chi_{AB}(i\omega_n) &= \chi_{\Theta^\dagger\hat{B}^\dagger\hat{\Theta},\Theta^\dagger\hat{A}^\dagger\Theta}(i\omega_n) = \epsilon_A^T\epsilon_B^T\chi_{B^\dagger A^\dagger}(i\omega_n)\end{aligned} \tag{2.87}$$

(for two operators with signatures ϵ_A^T and ϵ_B^T). It is easy to show that equations (2.85-2.87) also hold for the time-ordered correlation function $\chi_{AB}(t,t') = \langle 0|T\hat{A}(t)\hat{B}(t')|0\rangle$ and $\chi_{AB}(\tau,\tau') = \langle 0|T_\tau\hat{A}(\tau)\hat{B}(\tau')|0\rangle$ and at finite temperatures where the ground state expectation value $\langle 0|\cdots|0\rangle$ is replaced by the thermal average $\langle\cdots\rangle = \frac{1}{Z}\mathrm{Tr}(e^{-\beta\hat{H}}\cdots)$.

At finite temperatures, the correlation function $\chi_{AB}(t,t') = \mathrm{Tr}[\hat{\rho}\hat{A}(t)\hat{B}(t')]$ defined in (2.84) reads

$$\begin{aligned}\chi_{AB}(t,t') &= \sum_n(\langle n|\hat{\Theta}^\dagger)\hat{\rho}\hat{A}(t)\hat{B}(t')(\hat{\Theta}|n\rangle) \\ &= \sum_n\langle n|(\hat{\Theta}^\dagger\hat{\rho}\hat{A}(t)\hat{B}(t')\hat{\Theta})|n\rangle^* \\ &= \mathrm{Tr}[\hat{\rho}\hat{\Theta}^\dagger\hat{B}^\dagger(t')\hat{A}^\dagger(t)\hat{\Theta}],\end{aligned} \tag{2.88}$$

[15] $\hat{A}$ and $\hat{A}^\dagger$ have obviously the same signature ϵ_A^T under time reversal.

[16] The time-reversal invariance condition $\hat{H}=\hat{H}'$ ensures that the operator $\hat{A}(t) = e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t}$ in the Heisenberg representation has the same definition for $\mathcal{O}$ and $\mathcal{O}'$.

[17] This comes from $\hat{\Theta}^\dagger\hat{B}(\tau')^\dagger\hat{A}(\tau)^\dagger\hat{\Theta} = e^{-\tau'\hat{H}}\hat{\Theta}^\dagger\hat{B}\hat{\Theta}e^{(\tau'-\tau)\hat{H}}\hat{\Theta}^\dagger\hat{A}\hat{\Theta}e^{\tau\hat{H}}$ since $[\hat{A}(\tau)]^\dagger = \hat{A}^\dagger(-\tau)$ and $[\hat{B}(\tau)]^\dagger = \hat{B}^\dagger(-\tau)$.

where the last line is obtained using $[\hat\rho,\hat\Theta]=0$ in a time-reversal invariant system and the cyclic invariance of the trace ($\hat\rho=e^{-\beta\hat H}/Z$). Hence $\chi_{AB}(t,t')=\chi_{\Theta^\dagger\hat B^\dagger\hat\Theta,\Theta^\dagger\hat A^\dagger\Theta}(-t',-t)$, a property that can also be derived for the time-ordered correlation function $\chi_{AB}(t,t')=\mathrm{Tr}\{\hat\rho[T\hat A(t)\hat B(t')]\}$ and the imaginary-time correlation function $\chi_{AB}(\tau,\tau')=\mathrm{Tr}\{\hat\rho[T_\tau\hat A(\tau)\hat B(\tau')]\}$.

Spin-$\frac{1}{2}$ fermions

As an example we consider a spin-$\frac12$ particle, where $\hat{\mathbf{S}}=\hat{\boldsymbol\sigma}/2$ and

$$\hat\Theta=-i\hat\sigma^y\hat K. \tag{2.89}$$

Using

$$\begin{aligned}\hat\Theta|\mathbf{r},\sigma\rangle&=\sigma|\mathbf{r},-\sigma\rangle,\\ \hat\Theta|\mathbf{p},\sigma\rangle&=\sigma|-\mathbf{p},-\sigma\rangle,\end{aligned} \tag{2.90}$$

we find that the wave function of the time reversed state in the $(\mathbf{r},S^z)$ representation reads

$$\begin{aligned}\Psi'_\sigma(\mathbf{r},t)&=\langle\mathbf{r},\sigma|(\hat\Theta|\Psi(-t)\rangle)\\ &=\int d^dr'\sum_{\sigma'}\langle\mathbf{r},\sigma|(\hat\Theta|\mathbf{r}',\sigma'\rangle\langle\mathbf{r}',\sigma'|\Psi(-t)\rangle)\\ &=\int d^dr'\sum_{\sigma'}\sigma'\langle\mathbf{r},\sigma|\mathbf{r}',-\sigma'\rangle\Psi^*_{\sigma'}(\mathbf{r}',-t)=-\sigma\Psi^*_{-\sigma}(\mathbf{r},-t).\end{aligned} \tag{2.91}$$

In a many-particle system, $\hat\Theta$ is defined by

$$\hat\Theta|\mathbf{r}_1,\sigma_1;\cdots;\mathbf{r}_N,\sigma_N\}=\left(\prod_i\sigma_i\right)|\mathbf{r}_1,-\sigma_1;\cdots;\mathbf{r}_N,-\sigma_N\}. \tag{2.92}$$

As can be verified by a direct calculation, the annihilation and creation operators transform as

$$\begin{aligned}\hat\Theta^\dagger\hat\psi^{(\dagger)}_\sigma(\mathbf{r})\hat\Theta&=-\sigma\hat\psi^{(\dagger)}_{-\sigma}(\mathbf{r}),\\ \hat\Theta^\dagger\hat\psi^{(\dagger)}_\sigma(\mathbf{r},t)\hat\Theta&=-\sigma\hat\psi^{(\dagger)}_{-\sigma}(\mathbf{r},-t).\end{aligned} \tag{2.93}$$

One easily sees that the density operator $\hat n(\mathbf{r},t)=\sum_\sigma\hat\psi^\dagger_\sigma(\mathbf{r},t)\hat\psi_\sigma(\mathbf{r},t)$ is even under time reversal while the current operator $\hat{\mathbf{j}}(\mathbf{r},t)=-\frac{i}{2m}\sum_\sigma(\hat\psi^\dagger_\sigma(\mathbf{r},t)\boldsymbol\nabla\hat\psi_\sigma(\mathbf{r},t)-\text{c.c.})$ is odd. Thus their correlation functions transform as in (2.85) and (2.86), with $\epsilon^T_n=1$ and $\epsilon^T_j=-1$. As for the one-particle correlation function $G_{\sigma\sigma'}(\mathbf{r},t;\mathbf{r}',t')=-i\langle T\hat\psi_\sigma(\mathbf{r},t)\hat\psi^\dagger(\mathbf{r}',t')\rangle$, we obtain

$$G_{\sigma\sigma'}(\mathbf{r},t;\mathbf{r}',t')=\sigma\sigma'G_{\bar\sigma'\bar\sigma}(\mathbf{r}',-t';\mathbf{r},-t) \tag{2.94}$$

(and the analogous result in imaginary time) in a time-reversal invariant system. Similarly, the anomalous Green functions in a superconductor (Sec. 7.5) satisfy

$$F_{\sigma\sigma'}(\mathbf{r},t;\mathbf{r}',t')=-i\langle T\hat\psi_\sigma(\mathbf{r},t)\hat\psi_{\sigma'}(\mathbf{r}',t')\rangle=\sigma\sigma'F^\dagger_{\bar\sigma'\bar\sigma}(\mathbf{r}',-t';\mathbf{r},-t), \tag{2.95}$$

where $F^\dagger_{\sigma\sigma'}(\mathbf{r},t;\mathbf{r}',t')=-i\langle T\hat\psi^\dagger_\sigma(\mathbf{r},t)\hat\psi^\dagger_{\sigma'}(\mathbf{r}',t')\rangle$.

Bosons

Analogous results can be obtained for bosons. In particular, for spin-zero particles, $\hat\Theta$ is simply defined by $\hat\Theta|\mathbf{r}_1,\cdots,\mathbf{r}_N\} = |\mathbf{r}_1,\cdots,\mathbf{r}_N\}$, and the creation/annihilation operators transform as

$$\begin{aligned} \hat\Theta^\dagger \hat\psi^{(\dagger)}(\mathbf{r})\hat\Theta &= \hat\psi^{(\dagger)}(\mathbf{r}), \\ \hat\Theta^\dagger \hat\psi^{(\dagger)}(\mathbf{r},t)\hat\Theta &= \hat\psi^{(\dagger)}(\mathbf{r},-t). \end{aligned} \tag{2.96}$$

2.2.7 Gauge transformation

In the previous sections we have considered geometric symmetries, related to the structure of space and time. They are defined by a change of the space-time coordinates $\mathbf{r}\to\mathbf{r}'$, $\mathbf{p}\to\mathbf{p}'$, $\mathbf{S}\to\mathbf{S}'$, $t\to t'$. We discuss now an example pertaining to a different type of transformations, the so-called internal symmetries. These correspond to transformations that cannot be seen as resulting from a geometric transformation. One of the simplest examples is the global U(1) transformation, where the wavefunction changes by a mere phase factor,

Internal sym. OK? Gauge transf. are not symmetries stricto sensu!

$$\Psi'(\mathbf{r},t) = e^{i\Lambda}\Psi(\mathbf{r},t). \tag{2.97}$$

The corresponding Wigner operator is simply $e^{i\Lambda}$ times the identity operator and obviously commutes with the Hamiltonian.

When the particle couples to the electromagnetic field, the Schrödinger equation[18] $\hat H|\Psi\rangle = i\partial_t|\Psi\rangle$ is invariant in the gauge transformation

$$\begin{aligned} \Psi'(\mathbf{r},t) &= e^{iq\Lambda(\mathbf{r},t)}\Psi(\mathbf{r},t), \\ \mathbf{A}'(\mathbf{r},t) &= \mathbf{A}(\mathbf{r},t) + \boldsymbol\nabla\Lambda(\mathbf{r},t), \\ \phi'(\mathbf{r},t) &= \phi(\mathbf{r},t) - \partial_t\Lambda(\mathbf{r},t), \end{aligned} \tag{2.98}$$

where q is the electric charge and $(\phi,\mathbf{A})$ the scalar and vector potentials.[19] The Wigner operator is $\hat U = e^{iq\Lambda(\hat{\mathbf{r}},t)}$ and the invariance of the Hamiltonian [Eq. (2.16)] in the transformation (2.98) follows from $\hat U^\dagger\hat{\mathbf{r}}\hat U = \hat{\mathbf{r}}$ and $\hat U^\dagger\hat{\mathbf{p}}\hat U = \hat{\mathbf{p}} + q\boldsymbol\nabla\Lambda(\hat{\mathbf{r}},t)$.

Many-particle systems

In a many-particle system, the Wigner operator associated with a global U(1) transformation is $\hat U = e^{i\Lambda\hat N}$, and the field operator transforms as

$$\begin{aligned} \hat U^\dagger\hat\psi(\mathbf{r})\hat U &= e^{i\Lambda}\hat\psi(\mathbf{r}), \\ \hat U^\dagger\hat\psi^\dagger(\mathbf{r})\hat U &= e^{-i\Lambda}\hat\psi^\dagger(\mathbf{r}) \end{aligned} \tag{2.99}$$

(we ignore the particle spin). Invariance of the Hamiltonian in the transformation (2.99) implies that the total number of particles $\hat N$ (the generator of the transformation) is conserved: $[\hat N,\hat H]=0$ and $d\hat N_H/dt = 0$. We shall see in section 2.3.1 that the invariance in the global U(1) transformation also implies that the local density $\hat n(\mathbf{r},t)$ satisfies a continuity equation (Noether's theorem).

[18] $\hat H = \frac{1}{2m}[\hat{\mathbf{p}}^2 - q\hat{\mathbf{p}}\cdot\mathbf{A}(\hat{\mathbf{r}},t) - q\mathbf{A}(\hat{\mathbf{r}},t)\cdot\hat{\mathbf{p}} + q^2\mathbf{A}(\hat{\mathbf{r}},t)^2] + q\phi(\hat{\mathbf{r}},t) + V(\hat{\mathbf{r}})$.

[19] In quantum field theory, the procedure which consists in including gauge fields in order to promote a global invariance to a local one is known as the gauge principle.

In a (local) gauge transformation the field operator transforms as

$$\begin{aligned} \hat{U}^\dagger \hat{\psi}(\mathbf{r}) \hat{U} &= e^{iq\Lambda(\mathbf{r},t)} \hat{\psi}(\mathbf{r}), \\ \hat{U}^\dagger \hat{\psi}^\dagger(\mathbf{r}) \hat{U} &= e^{-iq\Lambda(\mathbf{r},t)} \hat{\psi}^\dagger(\mathbf{r}), \end{aligned} \tag{2.100}$$

where the Wigner operator is given by $\hat{U} = e^{iq\int d^d r\, \hat{n}(\mathbf{r})\Lambda(\mathbf{r},t)}$.

2.3 Symmetries and functional integrals

In this section, we start with classical field theory and Noether's theorem before discussing symmetries in the functional integral formalism. In particular we show that invariance in a continuous symmetry transformation implies a set of constraints for the Green functions and the vertex functions (Ward identities) and briefly discuss the consequences of spontaneous symmetry breaking (Goldstone theorem).

2.3.1 Classical field theory – Noether's theorem

We consider a classical system with the action[20]

$$S[\psi_j] = \int dt \int d^d r\, \mathcal{L}(\psi_j, \dot{\psi}_j, \boldsymbol{\nabla}\psi_j) \tag{2.101}$$

and the Lagrangian density $\mathcal{L}$, where the field $\psi_j(x)$ is a function of $x = (\mathbf{r}, t)$ and a discrete index j (which may include a spin index and/or refers to the real and imaginary parts of a complex field, etc.). The dynamics of the system is given by the Euler-Lagrange equations obtained from the minimization of the action,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_j(x)} = \frac{\partial \mathcal{L}}{\partial \psi_j(x)} - \boldsymbol{\nabla}\cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\psi_j(x)}. \tag{2.102}$$

In a symmetry transformation $\psi_j(x) \to \psi'_j(x)$, the Lagrangian density transforms into $\mathcal{L}'(\psi'_j, \dot{\psi}'_j, \boldsymbol{\nabla}\psi'_j)$. For instance, for a space translation, $\psi'_j(\mathbf{r}, t) = \psi_j(\mathbf{r}-\mathbf{a}, t)$. Since ψ_j and ψ'_j refer to the same system, one must have $S'[\psi'] = S[\psi]$ ($\mathcal{L}(\psi_j, \dot{\psi}_j, \boldsymbol{\nabla}\psi_j)$ and $\mathcal{L}'(\psi'_j, \dot{\psi}'_j, \boldsymbol{\nabla}\psi'_j)$ then coincide up to a total time or space derivative, see Sec. 1.A.3). If $S'[\psi'] = S[\psi']$, the equation of motion of $\psi'_j(x)$ is the same as that of $\psi_j(x)$ and the system is invariant in the transformation $\psi_j(x) \to \psi'_j(x)$.

Noether's theorem

Let us consider a continuous transformation, defined by

$$\psi_j(x) \to \psi_j(x) + i\epsilon F_j(x) \quad (\epsilon \to 0) \tag{2.103}$$

in its infinitesimal version, which leaves the action invariant. Here $F_j(x) \equiv F_j[x, \psi]$ is a function of x that depends functionally on ψ.[21] The Lagrangian density must be invariant up to a total derivative, i.e.

$$\mathcal{L}(\psi_j, \dot{\psi}_j, \boldsymbol{\nabla}\psi_j) \to \mathcal{L}(\psi_j, \dot{\psi}_j, \boldsymbol{\nabla}\psi_j) + \epsilon\partial_t \mathcal{F}_0(\psi_j) + \epsilon\boldsymbol{\nabla}\cdot \vec{\mathcal{F}}(\psi_j). \tag{2.104}$$

[20] Classical field theory is reviewed in Sec. 1.A.3.

[21] In general, $F_j(x)$ is a function of the field $\psi_{j'}(x)$ and its derivatives.

Noether's theorem states that for any continuous transformation, defined by (2.103) and (2.104), that leaves the action invariant, there is a density and a current,

$$\begin{aligned} j_0(x) &= -i \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_t \psi_j(x))} F_j(x) + \mathcal{F}_0(x), \\ \mathbf{j}(x) &= -i \sum_j \frac{\partial \mathcal{L}}{\partial(\boldsymbol{\nabla} \psi_j(x))} F_j(x) + \vec{\mathcal{F}}(x) \end{aligned} \tag{2.105}$$

($\mathcal{F}_\mu(x) \equiv \mathcal{F}_\mu(\psi_j(x))$) that satisfy a continuity equation (assuming that the fields satisfy the Euler-Lagrange equations)

$$\partial_t j_0(x) + \boldsymbol{\nabla} \cdot \mathbf{j}(x) = 0. \tag{2.106}$$

This implies the existence of a conserved (time-independent) charge,

$$Q = \int d^d r \, j_0(x), \qquad \frac{d}{dt} Q = 0 \tag{2.107}$$

(see Sec. 1.A.3 for a more detailed discussion).

What can we learn from Noether's theorem in quantum field theory?

Let us consider an infinitesimal continuous transformation where the operator $\hat{\psi}_j(\mathbf{r})$ in the quantum theory transforms as

$$\hat{\psi}_j(\mathbf{r}) \to \exp(-i\epsilon \hat{t}) \hat{\psi}_j(\mathbf{r}) \exp(i\epsilon \hat{t}) \simeq \hat{\psi}_j(\mathbf{r}) - i\epsilon[\hat{t}, \hat{\psi}_j(\mathbf{r})] \equiv \hat{\psi}_j(\mathbf{r}) + i\epsilon \hat{F}_j(\mathbf{r}). \tag{2.108}$$

$\hat{t}$ is the generator of the transformation. When the system is invariant under this transformation, $\hat{t}$ is conserved, i.e. $d\hat{t}_H/dt = 0$ in the Heisenberg picture (see Sec. 2.1).

In the Lagrangian formalism, the field $\psi_j(\mathbf{r}, t)$ transforms as

$$\psi_j(\mathbf{r}, t) \to \psi_j(\mathbf{r}, t) + i\epsilon F_j(\mathbf{r}, t), \tag{2.109}$$

where the dependence of $F_j(\mathbf{r}, t)$ on $\psi_j(\mathbf{r}, t)$ is obtained from that of $\hat{F}_j(\mathbf{r})$ on $\hat{\psi}_j(\mathbf{r})$ [Eq. (2.108)]. The conserved classical current is given by (2.105). For simplicity, we assume that $\mathcal{F}_0 = \vec{\mathcal{F}} = 0$ and that for any canonical field, i.e. any field satisfying $\partial \mathcal{L}/\partial \dot{\psi}_j(\mathbf{r}, t) \neq 0$, F_j does not depend on the conjugate momentum $\Pi_j(\mathbf{r}, t) = \partial \mathcal{L}/\partial \dot{\psi}_j(\mathbf{r}, t)$. We now quantize the theory and promote the classical (canonical) fields $\psi_j(\mathbf{r}, t)$ and their conjugate momenta $\Pi_j(\mathbf{r}, t)$ to operators satisfying the equal-time commutation relations $[\hat{\psi}_j(\mathbf{r}), \hat{\Pi}_{j'}(\mathbf{r}')] = i\delta_{j,j'}\delta(\mathbf{r} - \mathbf{r}')$. From the conserved classical current, we obtain the density operator

$$\hat{j}_0(\mathbf{r}) = -i \sum_j \hat{\Pi}_j(\mathbf{r}) \hat{F}_j(\mathbf{r}) \tag{2.110}$$

if we substitute the operators to the classical fields. Since by assumption $\hat{F}_j(\mathbf{r})$ does not depend on $\hat{\Pi}_j(\mathbf{r})$, we have

$$\begin{aligned} \int d^d r' [\hat{j}_0(\mathbf{r}'), \hat{\psi}_j(\mathbf{r})] &= -i \sum_{j'} \int d^d r' [\hat{\Pi}_{j'}(\mathbf{r}') \hat{F}_{j'}(\mathbf{r}'), \hat{\psi}_j(\mathbf{r})] \\ &= -i \sum_{j'} \int d^d r' [\hat{\Pi}_{j'}(\mathbf{r}'), \hat{\psi}_j(\mathbf{r})] \hat{F}_{j'}(\mathbf{r}') \\ &= -\hat{F}_j(\mathbf{r}) = [\hat{t}, \hat{\psi}_j(\mathbf{r})], \end{aligned} \tag{2.111}$$

where the last line is obtained using (2.108). We conclude that

$$\int d^d r\, \hat{j}_0(\mathbf{r}) = \hat{t}. \tag{2.112}$$

The charge $\hat{Q} = \int d^d r\, \hat{j}_0(\mathbf{r})$ is not only conserved but is also the generator of the symmetry transformation in the quantum theory. The invariance condition $d\hat{t}_H/dt = 0$ translates into

$$\frac{d}{dt} \int d^d r\, \hat{j}_0(\mathbf{r}, t) = 0, \tag{2.113}$$

where $\hat{j}_0(\mathbf{r}, t) = e^{i\hat{H}t} \hat{j}_0(\mathbf{r}) e^{-i\hat{H}t}$ is the current operator in the Heisenberg picture. In all cases of interest, one can show, using the equation of motion $\partial_t \hat{j}_0(\mathbf{r}, t) = i[\hat{H}, \hat{j}_0(\mathbf{r}, t)]$, that the current operator satisfies the continuity equation

$$\partial_t \hat{j}_0(\mathbf{r}, t) + \boldsymbol{\nabla} \cdot \hat{\mathbf{j}}(\mathbf{r}, t) = 0, \tag{2.114}$$

directly obtained from its classical counterpart [Eq. (2.106)] by substituting the operators to the fields.

Symmetries and invariances in the functional integral formalism

In section 2.2 we have seen how the invariance of a system in a symmetry transformation strongly constrains the correlation functions. This can also be seen in the functional integral formalism. Suppose we have identified a transformation $\psi_j(x) \to \psi'_j(x)$ which leaves the action invariant, $S[\psi'] = S[\psi]$, and has unit Jacobian. One then has[22]

$$\begin{aligned} \langle \psi'_{j_1}(x_1) \cdots \psi'_{j_n}(x_n) \rangle_{S[\psi]} &= \frac{1}{Z} \int \mathcal{D}[\psi]\, \psi'_{j_1}(x_1) \cdots \psi'_{j_n}(x_n) e^{-S[\psi]} \\ &= \frac{1}{Z} \int \mathcal{D}[\psi']\, \psi'_{j_1}(x_1) \cdots \psi'_{j_n}(x_n) e^{-S[\psi']} \\ &= \langle \psi_{j_1}(x_1) \cdots \psi_{j_n}(x_n) \rangle_{S[\psi]}. \end{aligned} \tag{2.115}$$

The second line is obtained by a change of variable in the functional integral using $S[\psi] = S[\psi']$ and $\mathcal{D}[\psi] = \mathcal{D}[\psi']$, while the last result follows from a mere relabeling of the field. Invariance thus implies that the transformed field $\psi'_j(x)$ have the same correlation functions as the original field $\psi_j(x)$. For example, for a particle system which is invariant in a parity transformation, $\psi'_\sigma(\mathbf{r}, \tau) = \psi_\sigma(-\mathbf{r}, \tau)$, equation (2.115) implies

$$\langle \psi_\sigma(-\mathbf{r}, \tau) \psi^*_{\sigma'}(-\mathbf{r}', \tau') \rangle = \langle \psi_\sigma(\mathbf{r}, \tau) \psi^*_{\sigma'}(\mathbf{r}', \tau') \rangle \tag{2.116}$$

Commentaire sur TRI?

(σ, σ' are spin indices), in agreement with (the Euclidean version of) equation (2.32).

2.3.2 Ward identities

In this section and the following ones we show that continuous transformations that leave the action invariance imply relations, known as Ward identities, between correlation functions or 1PI/2PI vertices.[23]

[22] From now on we consider the Matsubara (imaginary-time) formalism where $x = (\mathbf{r}, \tau)$. The field ψ_j can be a Grassmann variable.

[23] Ward identities can also be derived within the operator formalism starting from the continuity equation (2.114).

Let us consider the partition function $Z[J] = \int \mathcal{D}[\psi]\, e^{-S-S_J}$,[22] with

$$S = \int_0^\beta d\tau \int d^d r\, \mathcal{L}(\psi_j, \partial_\mu \psi_j),$$
$$S_J = -\int_0^\beta d\tau \int d^d r \sum_j J_j(x)\psi_j(x), \tag{2.117}$$

in the presence of an external source. $\mathcal{L}$ is now the Euclidean Lagrangian density. We assume that we have identified an infinitesimal transformation which leaves the measure of the functional integral and the action invariant [Eqs. (2.103) and (2.104)]. We then have[24]

$$\sum_j \left[iF_j(x) \frac{\partial \mathcal{L}}{\partial \psi_j(x)} + \sum_\mu i\partial_\mu F_j(x) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_j(x))} \right] = \sum_{\mu'} \partial_{\mu'} \mathcal{F}_{\mu'}(x), \tag{2.118}$$

where $\mu, \mu' = 0, x, y, \cdots$.

We now generalize the infinitesimal transformation by allowing ϵ to depend on x,

$$\psi_j(x) \to \psi_j(x) + i\epsilon(x) F_j(x). \tag{2.119}$$

The change in the action is

$$\begin{aligned} \delta S &= \int_0^\beta d\tau \int d^d r \sum_j \left\{ i\epsilon(x) F_j(x) \frac{\partial \mathcal{L}}{\partial \psi_j(x)} + \sum_\mu i\partial_\mu [\epsilon(x) F_j(x)] \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_j(x))} \right\} \\ &= \int_0^\beta d\tau \int d^d r \left\{ \epsilon(x) \sum_\mu \partial_\mu \mathcal{F}_\mu(x) + i \sum_{j,\mu} [\partial_\mu \epsilon(x)] F_j(x) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_j(x))} \right\}, \end{aligned} \tag{2.120}$$

where we have used (2.118). Integrating by part the last term in (2.120), we obtain

$$\begin{aligned} \delta S &= -\int_0^\beta d\tau \int d^d r\, \epsilon(x) \sum_\mu \partial_\mu \left[i \sum_j F_j(x) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_j(x))} - \mathcal{F}_\mu(x) \right] \\ &= -\int_0^\beta d\tau \int d^d r\, [i\partial_\tau j_0(x) + \boldsymbol{\nabla} \cdot \mathbf{j}(x)]\, \epsilon(x), \end{aligned} \tag{2.121}$$

where the current densities are defined by[25]

$$\begin{aligned} j_0(x) &= \sum_j F_j(x) \frac{\partial \mathcal{L}}{\partial(\partial_\tau \psi_j(x))} + i\mathcal{F}_0(x), \\ j_\mu(x) &= i \sum_j F_j(x) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_j(x))} - \mathcal{F}_\mu(x) \quad (\mu \neq 0). \end{aligned} \tag{2.122}$$

The transformation (2.119) is a change of variables which does not change the partition function $Z[J]$. If it leaves the measure invariant in the functional integral, we should have

$$\int \mathcal{D}[\psi]\, e^{-S-S_J} \int_0^\beta d\tau \int d^d r \left[i\partial_\tau j_0(x) + \boldsymbol{\nabla} \cdot \mathbf{j}(x) + i \sum_j J_j(x) F_j(x) \right] \epsilon(x) = 0 \tag{2.123}$$

[24] Note the position of the field $F_j(x)$ in (2.118) which follows from the chain rule (1.239) for derivation of Grassmann variables.

[25] The real-time currents (2.105) can be deduced from their imaginary-time counterparts (2.122) by the usual Wick rotation $-S \to iS$ and $\tau = it$. (Note that $\mathcal{L}$ in (2.122) is the Euclidean Lagrangian density.)

to lowest order in $\epsilon(x)$. Since this equation holds for an arbitrary function $\epsilon(x)$, we deduce

$$\int \mathcal{D}[\psi]\, e^{-S-S_J} \Big[i\partial_\tau j_0(x) + \boldsymbol{\nabla}\cdot\mathbf{j}(x) + i \sum_j J_j(x) F_j(x) \Big] = 0. \tag{2.124}$$

Taking the nth order functional derivative with respect to the source and putting $J_j(x) = 0$, we obtain

$$\begin{aligned} &\langle \psi_{j_n}(x_n)\cdots\psi_{j_1}(x_1)[i\partial_\tau j_0(x) + \boldsymbol{\nabla}\cdot\mathbf{j}(x)]\rangle = \\ &\quad - i \sum_{k=1}^n \langle \psi_{j_n}(x_n)\cdots\psi_{j_{k+1}}(x_{k+1}) F_{j_k}(x_k) \psi_{j_{k-1}}(x_{k-1})\cdots\psi_{j_1}(x_1)\rangle \delta(x-x_k). \end{aligned} \tag{2.125}$$

Note that the time derivatives acting on the fields can be pulled outside the functional integral in (2.124) which can then be written as a time-ordered correlation function in the operator formalism. For instance, assuming that $j_0(x)$ does not contain any time derivative, one has $\langle \psi_j(x_1)\partial_\tau j_0(x)\rangle \equiv \partial_\tau \langle T_\tau \hat\psi_j(x_1)\hat\jmath_0(x)\rangle$.[26] Equations (2.125) are exact relations between the correlation functions. These relations should be fulfilled in any theory which satisfies the continuity equation associated with the conservation law considered.

It is often convenient to write the Ward identities (2.125) in a different form starting from

$$S = -\frac{1}{2} \sum_{j,j'} \int_0^\beta d\tau d\tau' \int d^dr d^dr' \psi_j(x) G^{-1}_{0,jj'}(x,x') \psi_{j'}(x') + S_{\rm int}. \tag{2.126}$$

We assume that $S_{\rm int}$ is invariant under the local symmetry transformation $\psi_j(x) \to \psi_j(x) + i\epsilon(x)F_j(x)$, which means that the Noether currents (2.122) depends only on S_0 (compare (2.127) below with (2.121)). The change in the action is then

$$\delta S = \delta S_0 = -i \sum_{j,j'} \int_0^\beta d\tau d\tau' \int d^dr d^dr' \epsilon(x) F_j(x) G^{-1}_{0,jj'}(x,x')\psi_{j'}(x'), \tag{2.127}$$

where we have used $G_{0,jj'}(x,x') = \zeta G_{0,j'j}(x',x)$ where $\zeta = 1$ (-1) for bosons (fermions). Since $Z[J]$ is unaltered in the change of variable $\psi_j(x) \to \psi(x) + \epsilon(x)F_j(x)$,

$$\int \mathcal{D}[\psi] e^{-S-S_J} \Big[\int_0^\beta d\tau' \int d^dr' \sum_{j,j'} F_j(x) G^{-1}_{0,jj'}(x,x')\psi_{j'}(x') + \sum_j J_j(x) F_j(x) \Big] = 0. \tag{2.128}$$

Taking the nth functional derivative with respect to the source and setting $J_j(x) = 0$, we obtain an alternative form of the Ward identities,

$$\begin{aligned} &\int_0^\beta d\tau' \int d^dr' \sum_{j,j'} \langle \psi_{j_n}(x_n)\cdots\psi_{j_1}(x_1) F_j(x)\psi_{j'}(x')\rangle G^{-1}_{0,jj'}(x,x') = \\ &\quad - \sum_{k=1}^n \langle \psi_{j_n}(x_n)\cdots\psi_{j_{k+1}}(x_{k+1}) F_{j_k}(x_k) \psi_{j_{k-1}}(x_{k-1})\cdots\psi_{j_1}(x_1)\rangle \delta(x-x_k). \end{aligned} \tag{2.129}$$

Unlike (2.125), equation (2.129) does not assume that the action can be written in terms of a Lagrangian density $\mathcal{L}(\psi_j, \partial_\mu\psi_j)$.

[26] When one has a certain identity in the functional integral formalism, one must pulled the time derivatives outside the functional integral before writing the identity in terms of time-ordered correlation functions in the operator formalism.

Spontaneously broken symmetry and Goldstone theorem

So far we have assumed that the limit $J \to 0$ can be taken with no difficulty. This is not the case when the symmetry is spontaneously broken. To see this let us consider a global symmetry transformation ($\epsilon(x) = \epsilon$) and assume that $F_j(x)$ is a linear function of $\psi_j(x)$,

$$F_j(x) = \int_0^\beta d\tau' \int d^d r' \sum_{j'} t_{jj'}(x, x')\psi_{j'}(x'). \tag{2.130}$$

Since $\delta S_0 = 0$ when $\epsilon(x) = \epsilon$,

$$\begin{aligned} 0 &= \int \mathcal{D}[\psi] e^{-S-S_J} \int_0^\beta d\tau \int d^d r \sum_j J_j(x) F_j(x) \\ &= \int_0^\beta d\tau d\tau' \int d^d r d^d r' \sum_{j,j'} J_j(x) t_{jj'}(x, x') \langle \psi_{j'}(x')\rangle_J \\ &= \int_0^\beta d\tau d\tau' \int d^d r d^d r' \sum_{j,j'} J_j(x) t_{jj'}(x, x') \frac{\delta \ln Z[J]}{\delta J_{j'}(x')}. \end{aligned} \tag{2.131}$$

Taking the functional derivative with respect to $J_k(y)$, we obtain

$$\begin{aligned} &\int_0^\beta d\tau' \int d^d r' \sum_{j'} t_{kj'}(y, x') \langle \psi_{j'}(x')\rangle_J \\ &+ \int_0^\beta d\tau d\tau' \int d^d r d^d r' \sum_{j,j'} J_j(x) t_{jj'}(x, x') \frac{\delta^2 \ln Z[J]}{\delta J_k(y) \delta J_{j'}(x')} = 0. \end{aligned} \tag{2.132}$$

Let us suppose that $\langle \psi_j(x)\rangle \equiv \langle \psi_j(x)\rangle_{J=0} \neq 0$ (which is possible only if ψ is a bosonic field). If $\int_0^\beta d\tau' \int d^d r' \sum_{j'} t_{kj'}(y, x')\langle \psi_{j'}(x')\rangle$ is nonzero, the symmetry is spontaneously broken: although the action is invariant in the transformation defined by (2.103) and (2.130), the physical system is not since $\langle \psi_k(y)\rangle \to \langle \psi_k(y)\rangle + i\epsilon \int d^d r' \int_0^\beta d\tau' \sum_{j'} t_{kj'}(y, x')\langle \psi_{j'}(x')\rangle \neq \langle \psi_k(y)\rangle$. In that case the second term in the lhs of (2.132) must remain finite as $J \to 0$. This is possible only if the response function

$$\int_0^\beta d\tau' \int d^d r' \sum_{j'} t_{jj'}(x, x') \frac{\delta^2 \ln Z[J]}{\delta J_k(y) \delta J_{j'}(x')}\bigg|_{J=0} \tag{2.133}$$

diverges. Such a divergence signals the existence of excitations with vanishing energies and is at the root of the Goldstone theorem (see Secs. 2.3.3 below and 3.6.3).

2.3.3 Ward identities for the 1PI vertices

Symmetries of the effective action $\Gamma[\phi]$

The effective action $\Gamma[\phi]$ is defined as the Legendre transform

$$\begin{aligned} \Gamma[\phi] &= -\ln Z[J] + \int_0^\beta d\tau \int d^d r \sum_j J_j(x)\phi_j(x), \\ \phi_j[x; J] &= \langle \psi_j(x)\rangle_J = \frac{\delta \ln Z[J]}{\delta J_j(x)}, \end{aligned} \tag{2.134}$$

of the generating functional $\ln Z[J]$ of connected Green functions (Sec. 1.6.2). In this section, we shall need two important properties of the effective action,[27]

$$\begin{aligned} \frac{\delta\Gamma[\phi]}{\delta\phi_j(x)} &= \zeta J_j(x), \\ \frac{\delta^2\Gamma[\phi]}{\delta\phi_{j_1}(x_1)\delta\phi_{j_2}(x_2)}\bigg|_{\phi=\bar\phi} &= \zeta\Gamma^{(2)}_{j_1j_2}(x_1,x_2) = -\zeta G^{-1}_{j_1j_2}(x_1,x_2), \end{aligned} \tag{2.135}$$

where $G_{j_1j_2}(x_1,x_2) = -\langle\psi_{j_1}(x_1)\psi_{j_2}(x_2)\rangle + \langle\psi_{j_1}(x_1)\rangle\langle\psi_{j_2}(x_2)\rangle$ is the connected two-point Green function.[28] We denote by $\bar\phi_j(x) = \langle\psi_j(x)\rangle_{J=0}$ the expectation value of $\psi_j(x)$ in the absence of the external source ($\bar\phi_j$ always vanishes for a fermionic field ψ_j).

Global Ward identities and Goldstone theorem

Let us first consider a global symmetry transformation ($\epsilon(x) = \epsilon$) defined by (2.119) and (2.130), which leaves the partition function invariant when $J = 0$. Equation (2.131) can be rewritten as

$$\int_0^\beta d\tau d\tau' \int d^d r d^d r' \sum_{j,j'} t_{jj'}(x,x')\phi_{j'}(x')\frac{\delta\Gamma[\phi]}{\delta\phi_j(x)} = 0. \tag{2.136}$$

The effective action $\Gamma[\phi]$ is therefore invariant under the infinitesimal transformation

$$\phi_j(x) \to \phi_j(x) + i\epsilon\langle F_j(x)\rangle = \phi_j(x) + i\epsilon\int_0^\beta d\tau'\int d^d r'\sum_{j'} t_{jj'}(x,x')\phi_{j'}(x'). \tag{2.137}$$

We conclude that infinitesimal linear transformations that leave both the action $S[\psi]$ and the measure of the functional integral invariant also leave the effective action $\Gamma[\phi]$ invariant.

Taking the functional derivative with respect to $\phi_k(y)$ in (2.136) and setting $J = 0$ (i.e. $\phi = \bar\phi$), we obtain

$$\int_0^\beta d\tau d\tau' \int d^d r d^d r' \sum_{j,j'} t_{jj'}(x,x')\bar\phi_{j'}(x')\Gamma^{(2)}_{kj}(y,x) = 0. \tag{2.138}$$

Similarly by taking two functional derivatives, we obtain

$$\begin{aligned} \int_0^\beta d\tau \int d^d r \sum_j \bigg\{ & t_{jk}(x,y)\Gamma^{(2)}_{jl}(x,z) + t_{jl}(x,z)\Gamma^{(2)}_{kj}(y,x) \\ & + \int_0^\beta d\tau' \int d^d r' \sum_{j'} t_{jj'}(x,x')\bar\phi_{j'}(x')\Gamma^{(3)}_{jkl}(x,y,z) \bigg\} = 0. \end{aligned} \tag{2.139}$$

(we have used the definition of the vertices given in the footnote 27). Equation (2.139), which relates $\Gamma^{(2)}$ and $\Gamma^{(3)}$, is an example of a (global) Ward identity.[29]

[27] Recall that the n-point 1PI vertices are defined by $\Gamma^{(n)}_{j_1\cdots j_n}(x_1,\cdots,x_n) = \frac{\delta^n\Gamma[\phi]}{\delta\phi_{j_n}(x_n)\cdots\delta\phi_{j_1}(x_1)}\Big|_{\phi=\bar\phi}$ (see Sec. 1.6.2).

[28] Note that the sign convention of the Green function $G_{j_1j_2}(x_1,x_2)$ differs from that used in Sec. 1.6.2: $G(\tilde\alpha\tau,\tilde\alpha'\tau') = \langle\psi_{\tilde\alpha}(\tau)\psi_{\tilde\alpha'}(\tau')\rangle - \langle\psi_{\tilde\alpha}(\tau)\rangle\langle\psi_{\tilde\alpha'}(\tau')\rangle$.

[29] This Ward identity is global to the extent where it involves an integration over space (in Fourier space, its typically involves zero-momentum vertices).

Equation (2.138) yields

$$\int_0^\beta d\tau d\tau' \int d^d r d^d r' \sum_{j,j'} G^{-1}_{kj}(y,x) t_{jj'}(x,x') \bar\phi_{j'}(x') = 0. \tag{2.140}$$

For simplicity, we assume that the system is translation invariant: $\bar\phi_j(x) = \bar\phi_j$, $t_{jj'}(x,x') = t_{jj'}(x-x')$ and $G_{jj'}(x,x') = G_{jj'}(x-x')$. Equation (2.140) then gives

$$\sum_{j,j'} G^{-1}_{kj}(q=0) t_{jj'}(q=0)\bar\phi_{j'} = 0, \tag{2.141}$$

where $q = (\mathbf{q}, i\omega_n)$ with ω_n a Matsubara frequency. This equation is trivially satisfied when $\sum_{j'} t_{jj'}(q=0)\bar\phi_{j'} = 0$. When $\sum_{j'} t_{jj'}(q=0)\bar\phi_{j'} \neq 0$, the state of the system is not invariant under the symmetry transformation; there is spontaneous symmetry breaking. Equation (2.141) then shows that $\sum_{j'} t_{jj'}(q=0)\bar\phi_{j'}$ is an eigenvector of the matrix $G^{-1}_{kj}(q=0)$ with a vanishing eigenvalue. In other words, there is a zero momentum excitation with vanishing energy. This result is truly interesting only if it implies that there is an excitation mode with an energy $\omega_{\mathbf{q}}$ which vanishes in the limit $\mathbf{q} \to 0$. This turns out to be the case, provided that the interactions in the system are short range. We shall come back to this issue in section 3.6.3 where the Goldstone theorem will be discussed from a different point of view.

Local Ward identities

More generally, Ward identities for the 1PI vertices are derived by considering a local symmetry transformation. The invariance of the partition function $Z[J]$ in the infinitesimal transformation (2.119) implies (2.128), which can be rewritten as

$$\sum_{j_1} \Bigl\langle \sum_{1'} F(1) G_0^{-1}(1,1')\psi(1') + J(1)F(1) \Bigr\rangle = 0 \tag{2.142}$$

using the compact notation $1 = (j_1, \mathbf{r}_1, \tau_1)$ and $\sum_1 = \int_0^\beta d\tau_1 \int d^d r_1 \sum_{j_1}$. For the linear transformation (2.130),

$$\begin{aligned} 0 &= \sum_{j_1,1''} t(1,1'') \Bigl\langle \sum_{1'} \psi(1'') G_0^{-1}(1,1')\psi(1') + J(1)\psi(1'') \Bigr\rangle \\ &= \sum_{j_1,1''} t(1,1'') \biggl\{ \sum_{1'} G_0^{-1}(1,1') \left(\frac{\delta^2 \ln Z[J]}{\delta J(1'')\delta J(1')} + \frac{\delta \ln Z[J]}{\delta J(1'')} \frac{\delta \ln Z[J]}{\delta J(1')} \right) + J(1) \frac{\delta \ln Z[J]}{\delta J(1'')} \biggr\}. \end{aligned} \tag{2.143}$$

Taking the functional derivative with respect to $\phi(2)$ and $\phi(3)$ and setting $J=0$, we obtain

$$\begin{aligned} &\sum_{j_1,1''} t(1,1'') \biggl\{ \sum_{1'} G_0^{-1}(1,1') \Bigl[-\sum_{2',3'} \Gamma^{(2)-1}(1'',2')\Gamma^{(4)}(3',2',2,3)\Gamma^{(2)-1}(3',1') \\ &+ \delta(1'',2)\delta(1',3) + \zeta\delta(1'',3)\delta(1',2) \Bigr] + \delta(1'',2)\Gamma^{(2)}(1,3) + \delta(1'',3)\Gamma^{(2)}(2,1) \biggr\} = 0 \end{aligned} \tag{2.144}$$

assuming $\bar\phi = 0$. Using $\Gamma^{(2)} = -G^{-1}$ and $G^{-1} = G_0^{-1} - \Sigma$, we can rewrite (2.144) as

$$\sum_{j_1,1',1'',2',3'} t(1,1'')G_0^{-1}(1,1')G(1',2')\Gamma^{(4)}(3,2,2',3')G(3',1'') = \sum_{j_1}[t(1,2)\Sigma(1,3) + t(1,3)\Sigma(2,1)]. \tag{2.145}$$

Equation (2.145) is a Ward identity relating the self-energy Σ to the two-particle vertex $\Gamma^{(4)}$. It is straightforward to extend the derivation to the case where $\bar\phi \neq 0$. Higher-order Ward identities can be obtained by considering additional functional derivatives with respect to ϕ before setting $J = 0$.

Example 1: Superfluid boson system

Let us consider an interacting boson system. Conservation of particles implies the action is invariant under the infinitesimal U(1) transformation $\psi(x) \to \psi(x) + i\epsilon\psi(x)$, $\psi^*(x) \to \psi^*(x) - i\epsilon\psi^*(x)$.[30] This linear transformation also leaves the effective action $\Gamma[\phi^*, \phi]$ invariant. From (2.141), we deduce

$$G_c^{-1}(q=0)\begin{pmatrix} \bar\phi \\ -\bar\phi^* \end{pmatrix} = 0, \tag{2.146}$$

where $\bar\phi = \langle\psi(x)\rangle$, $\bar\phi^* = \langle\psi^*(x)\rangle$, and

$$G_c(x-x') = -\begin{pmatrix} \langle\psi(x)\psi(x')\rangle - \bar\phi^2 & \langle\psi(x)\psi^*(x')\rangle - |\bar\phi|^2 \\ \langle\psi^*(x)\psi(x')\rangle - |\bar\phi|^2 & \langle\psi^*(x)\psi^*(x')\rangle - (\bar\phi^*)^2 \end{pmatrix} \tag{2.147}$$

is the single-particle Green function (written here as a 2×2 matrix). If $|\bar\phi| \neq 0$, the U(1) symmetry is spontaneously broken. We shall see in chapter 7 that this property is characteristic of superfluid systems. The Goldstone mode, whose existence can be inferred from (2.146) when the interactions are short range, is the Bogoliubov sound mode.

Example 2: Magnetism in localized spin systems

Let us consider quantum spins located at the sites $\mathbf{r}$ of a three-dimensional cubic lattice. The partition function can be written as a spin coherent-state functional integral with the action $S[\mathbf{\Omega}]$, where $\mathbf{\Omega}_\mathbf{r}$ is a unit vector field ($\mathbf{\Omega}_\mathbf{r}^2 = 1$) (see Sec. 8.4.1). As usual, the effective action $\Gamma[\mathbf{M}]$ is defined as the Legendre transform of $\ln Z[\mathbf{h}]$ where $\mathbf{h}$ is an external field which couples to the spins. Assuming spin rotation invariance in the absence of external field, $\Gamma[\mathbf{M}]$ is invariant under an infinitesimal rotation about an arbitrary axis $\mathbf{n}$, $\mathbf{M} \to \mathbf{M} + \epsilon\mathbf{n}\times\mathbf{M}$. Equation (2.140) gives

$$\int_0^\beta d\tau\, d\tau' \sum_{\substack{\mathbf{r},\mathbf{r}'\\ i,j,k}} G_{li}^{-1}(x,x')\epsilon_{ijk}n_j\bar M_k(x') = 0, \tag{2.148}$$

where $\bar{\mathbf{M}}(x) = \langle\mathbf{\Omega}(x)\rangle_{\mathbf{h}=0}$ is the zero-field magnetization, and

$$G_{li}(x,x') = \langle\Omega_l(x)\Omega_i(x')\rangle - \langle\Omega_l(x)\rangle\langle\Omega_i(x')\rangle \tag{2.149}$$

[30] Obviously the U(1) transformation leaves the measure in the functional integral invariant.

the spin-spin correlation function.

If the system is ferromagnetic, $\bar{M}_k(x) = \delta_{k,z}\bar{M}_0$ (assuming the magnetization along the z direction), we obtain

$$\sum_{i,j} G^{-1}_{li}(q=0)\epsilon_{ijz}n_j = 0. \tag{2.150}$$

This equation is trivially satisfied for a rotation about the z axis, since $\epsilon_{izz} = 0$. By considering rotations about the x or y axis, we conclude that $G^{-1}_{yy}(q=0)$ and $G^{-1}_{xx}(q=0)$ vanish.[31] The gapless excitations showing up as poles of $G_{xx}(q)$ and $G_{yy}(q)$ are (ferromagnetic) spin waves. They will be discussed in detail in chapter 8.

If the system exhibits antiferromagnetic long-range order, $\bar{M}_k(x) = \delta_{k,z}e^{i\mathbf{Q}\cdot\mathbf{r}}\bar{M}_0$ with $\mathbf{Q} = (\pi,\pi,\pi)$. Equation (2.148) gives

$$\sum_{i,j} G^{-1}_{li}(q=Q)\epsilon_{ijz}n_j = 0, \tag{2.151}$$

where $Q = (\mathbf{Q}, 0)$. Both $G^{-1}_{yy}(Q)$ and $G^{-1}_{xx}(Q)$ vanish. The Goldstone modes are antiferromagnetic spin waves (chapter 8).

2.3.4 Ward identities for the 2PI vertices

In the 2PI scheme (Sec. 1.6.3), the quantity to be considered is the functional

$$\begin{aligned}\Gamma[G] &= -\ln Z[J] - \frac{1}{2}\int d\tau_1 d\tau_2 \int d^dr_1 d^dr_2 \sum_{j_1,j_2} J_{j_1j_2}(x_1,x_2)G_{j_1j_2}(x_1,x_2) \\ &= -\frac{\zeta}{2}\mathrm{Tr}\ln(-G) + \frac{\zeta}{2}\mathrm{Tr}(G_0^{-1}G - 1) + \Phi[G].\end{aligned} \tag{2.152}$$

Here $J_{j_1j_2}(x_1,x_2) = \zeta J_{j_2j_1}(x_2,x_1)$ is a bilinear external source which couples to $\psi_{j_1}(x_1)\psi_{j_2}(x_2)$ and G is the one-particle Green function.[28]

$$\begin{aligned}G_{j_1j_2}[x_1,x_2;J] &= -\langle\psi_{j_1}(x_1)\psi_{j_2}(x_2)\rangle_J = -\frac{\delta\ln Z[J]}{\delta J_{j_1j_2}(x_1,x_2)}, \\ \frac{\delta\Gamma[G]}{\delta G_{j_1j_2}(x_1,x_2)} &= -J_{j_1j_2}(x_1,x_2).\end{aligned} \tag{2.153}$$

The actual Green function $\bar{G}$, corresponding to a vanishing external source, is obtained from the stationary point of $\Gamma[G]$: $\delta\Gamma/\delta G|_{G=\bar{G}} = \Gamma^{(1)} = 0$. The Luttinger-Ward functional $\Phi[G]$ is the generating functional of the 2PI vertices $\Phi^{(n)}$.

Global Ward identities and Goldstone theorem

Let us first consider a global linear transformation defined by (2.103) and (2.130),

$$\psi_j(x) \to \psi_j(x) + i\epsilon\int_0^\beta d\tau' \int d^dr' \sum_{j'} t_{jj'}(x')\psi_{j'}(x') \tag{2.154}$$

[31] For symmetry reasons, $G_{xy}(q=0)$ always vanishes.

which leaves the action $S_0 + S_{\rm int}$ invariant. Proceeding as in section 2.3.3 and using the same compact notation, one easily finds that

$$\sum_{1,2,3} \frac{\delta\Gamma[G]}{\delta G(1,2)} t(1,3)G(3,2) = 0. \tag{2.155}$$

Equation (2.155) expresses the invariance of $\Gamma[G]$ in the transformation

$$G(1,2) \to G(1,2) + i\epsilon \sum_3 [t(1,3)G(3,2) + t(2,3)G(1,3)], \tag{2.156}$$

which follows from (2.154). Taking the functional derivative with respect to $G(1',2')$ in (2.155) and setting $J = 0$, we obtain the (global) Ward identity

$$\sum_{1,2,3} \frac{\delta^2\Gamma[G]}{\delta G(1',2')\delta G(1,2)}\bigg|_{G=\bar G} [t(1,3)\bar G(3,2) + t(2,3)\bar G(1,3)] = 0. \tag{2.157}$$

If $\sum_3 [t(1,3)\bar G(3,2) + t(2,3)\bar G(1,3)] \neq 0$, the state of the system is not invariant under the symmetry transformation (2.156): the symmetry is spontaneously broken. $\delta^2\Gamma/\delta G\delta G|_{\bar G} = W^{(2)-1}$ must then have a vanishing eigenvalue.[32] We thus reproduce Goldstone theorem. Additional Ward identities can be obtained from (2.155) by taking further functional derivatives.

Symmetry properties of $\Phi[G]$ and local Ward identities

We now consider a local symmetry transformation defined by (2.154) but with $\epsilon \to \epsilon(x)$, which leaves $S_{\rm int}$ invariant. The corresponding change in the Green function is

$$G(1,2) \to G(1,2) + i \sum_3 [\epsilon(x_1)t(1,3)G(3,2) + \epsilon(x_2)t(2,3)G(1,3)]. \tag{2.158}$$

The invariance of the partition function implies

$$\sum_{j_1,2,3} \left[G_0^{-1}(1,2) - \frac{\delta\Gamma[G]}{\delta G(1,2)} \right] t(1,3)G(3,2) = 0. \tag{2.159}$$

From (2.158) and (2.159), we find that the functional $\Gamma[G]$ varies by[33]

$$\begin{aligned} \delta\Gamma[G] &= \frac{i}{2} \sum_{1,2,3} \frac{\delta\Gamma[G]}{\delta G(1,2)} [\epsilon(x_1)t(1,3)G(3,2) + \epsilon(x_2)t(2,3)G(1,3)] \\ &= \frac{i}{2} \sum_{1,2,3} G_0^{-1}(1,2) [\epsilon(x_1)t(1,3)G(3,2) + \epsilon(x_2)t(2,3)G(1,3)] \end{aligned} \tag{2.160}$$

to leading order in $\epsilon(x)$. Comparing this result with the definition (2.152) of $\Gamma[G]$, we see that the change in Γ is entirely due to $\frac{\zeta}{2}{\rm Tr}(G_0^{-1}G - 1)$, which implies that $-\frac{\zeta}{2}{\rm Tr}\ln(-G) + \Phi[G]$ is invariant in the transformation (2.158). Now, since $\Phi[G]$ vanishes in the absence of interaction, $-\frac{\zeta}{2}{\rm Tr}\ln(-G)$ must be invariant. We conclude that the Luttinger-Ward functional is invariant in any infinitesimal transformation that leaves $S_{\rm int}$ invariant.

[32] $W^{(2)}(1,2;1',2') = \langle\psi(1)\psi(2)\psi(1')\psi(2')\rangle - \langle\psi(1)\psi(2)\rangle\langle\psi(1')\psi(2')\rangle$ (see Sec. 1.6.3).
[33] The reason for the factor 1/2 in (2.160) is explained in Sec. 1.6.3 (page 90).

The invariance of $\Phi[G]$ can also be understood in a diagrammatic picture. As an example, let us consider a local U(1) transformation, $\psi(x) \to e^{i\epsilon(x)}\psi(x)$ and $\psi^*(x) \to e^{-i\epsilon(x)}\psi^*(x)$, whereby the Green function transforms as

$$G(x,x') \to e^{\pm i\epsilon(x)} G(x,x') e^{\pm i\epsilon(x')}. \tag{2.161}$$

For any skeleton diagram contributing to $\Phi[G]$, each vertex associated with $v(x_1,x_2,x_3,x_4)\psi^*(x_1)\psi^*(x_2)\psi(x_3)\psi(x_4)$ picks up a factor $\exp\{i[-\epsilon(x_1)-\epsilon(x_2)+\epsilon(x_3)+\epsilon(x_4)]\}$. But the invariance of $S_{\rm int}$ in the local U(1) transformation requires this phase factor to be unity, which implies the invariance of any diagram contributing to $\Phi[G]$. A similar reasoning can be made for other symmetry transformations that leave $S_{\rm int}$ invariant.

Local Ward identities for the 2PI vertices follow from the symmetry properties of the Luttinger-Ward functional. The invariance of $\Phi[G]$ in the transformation (2.158) implies

$$\sum_{j_1,2,3} \frac{\delta\Phi[G]}{\delta G(1,2)} t(1,3) G(3,2) = 0. \tag{2.162}$$

Taking the functional derivative with respect to $G(1',2')$ and setting $J=0$, i.e. $G=\bar G$, we obtain

$$\sum_{j_1,2,3} \Phi^{(2)}(1',2';1,2) t(1,3)\bar G(3,2) = \sum_{j_1} [t(1,1')\Sigma(1,2') + t(1,2')\Sigma(1',1)], \tag{2.163}$$

where $\Sigma = -\Phi^{(1)}$. Equation (2.163) is a Ward identity relating the self-energy Σ to the 2PI two-particle vertex $\Phi^{(2)}$. Higher-order Ward identities can be obtained by taking additional functional derivatives.

2.A Ward identities in a fermion system

In this appendix, we use the results of section 2.3.3 to derive Ward identities in a spin-$\frac{1}{2}$ normal (i.e. nonsuperfluid) fermion system.[34] We start from equation (2.145) relating the self-energy to the 1PI two-particle vertex $\Gamma^{(4)}$. The index j includes a spin index $\sigma = \uparrow,\downarrow$ and a charge index $c=\pm$ such that $\psi_j = \psi_\sigma$ (ψ^*_σ) if $c = -$ (+). We assume that the system is not superconducting (i.e. the U(1) symmetry is not broken) and the propagators diagonal in spin. The latter assumption does not eliminate a broken spin-rotation symmetry state but restricts the possible states of the system.[35] We further assume that

$$t_{jj'}(x,x') = \begin{cases} t^{(+)}_{\sigma\sigma'}(x,x') & \text{if} \quad c=c'=+ \\ t^{(-)}_{\sigma\sigma'}(x,x') & \text{if} \quad c=c'=- \\ 0 & \text{if} \quad c\neq c' \end{cases} \tag{2.164}$$

Equation (2.145) then gives

$$\begin{aligned} \int_0^\beta d\tau_{1'} d\tau_{1''} d\tau_{2'} d\tau_{3'} \int d^dr_{1'} d^dr_{1''} d^dr_{2'} d^dr_{3'} \sum_{\sigma_1,\sigma_{1''}} \Big[t^{(-)}_{\sigma_{1''}\sigma_1}(x_1,x_{1'}) G^{-1}_{0,\sigma_{1''}}(x_{1''},x_1) \\ + t^{(+)}_{\sigma_1\sigma_{1''}}(x_1,x_{1''}) G^{-1}_{0,\sigma_1}(x_1,x_{1'}) \Big] G_{\sigma_1}(x_{1'},x_{2'}) \Gamma^{(4)}_{{\rm ph},\sigma_3\sigma_2\sigma_1\sigma_{1''}}(x_3,x_2,x_{2'},x_{3'}) G_{\sigma_{1''}}(x_{3'},x_{1''}) \\ = -t^{(-)}_{\sigma_3\sigma_2}(x_1,x_2)\Sigma_{\sigma_3}(x_3,x_1) - t^{(+)}_{\sigma_2,\sigma_3}(x_1,x_3)\Sigma_{\sigma_2}(x_1,x_2), \end{aligned} \tag{2.165}$$

[34] These Ward identities play a crucial role in Fermi-liquid theory (chapter 4).
[35] For instance, if the system is ferromagnetic or antiferromagnetic, the magnetization must be along the spin-quantization axis.

where

$$\Gamma^{(4)}_{\mathrm{ph},\sigma_1\sigma_{1'}\sigma_2\sigma_{2'}}(x_1,x_{1'},x_2,x_{2'}) \equiv \Gamma^{(4)}_{\sigma_1\sigma_{1'}\sigma_2\sigma_{2'}}(x_1+,x_{1'}-,x_2+,x_{2'}-) \tag{2.166}$$

is the two-particle vertex in the particle-hole channel. Introducing the Fourier transform

$$t^{(\pm)}(k) = \int_0^\beta d\tau \int d^d r\, e^{\pm i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)} t^{(\pm)}(x), \tag{2.167}$$

and multiplying (2.165) by $\frac{1}{\beta V}\int_0^\beta d\tau_1 d\tau_2 d\tau_3 \int d^d r_1 d^d r_2 d^d r_3 e^{iqx_1+ikx_2+ik_3x_3}$, we obtain

$$\begin{aligned}\frac{1}{\beta V}\sum_{k',\sigma,\sigma'} &\left[t^{(-)}_{\sigma'\sigma}(k')G^{-1}_{0,\sigma'}(k'+q) + t^{(+)}_{\sigma\sigma'}(k'+q)G^{-1}_{0,\sigma}(k')\right] G_\sigma(k')G_{\sigma'}(k'+q) \\ &\times \Gamma^{(4)}_{\mathrm{ph},\sigma_3\sigma_2\sigma\sigma'}(k+q,k,k',k'+q) = -t^{(-)}_{\sigma_3\sigma_2}(k)\Sigma_{\sigma_3}(k+q) - t^{(+)}_{\sigma_2\sigma_3}(k+q)\Sigma_{\sigma_2}(k).\end{aligned} \tag{2.168}$$

Gauge transformation

For a gauge transformation,

$$\begin{aligned}\psi_\sigma(x) &\to \psi_\sigma(x)e^{i\epsilon(x)} = \psi_\sigma(x) + i\epsilon(x)\psi_\sigma(x) + \mathcal{O}(\epsilon^2), \\ \psi^*_\sigma(x) &\to \psi^*_\sigma(x)e^{-i\epsilon(x)} = \psi^*_\sigma(x) - i\epsilon(x)\psi^*_\sigma(x) + \mathcal{O}(\epsilon^2),\end{aligned} \tag{2.169}$$

we have $t^{(\pm)}_{\sigma\sigma'}(x,x') = \mp\delta_{\sigma,\sigma'}\delta(x-x')$. The corresponding Ward identity reads

$$\begin{aligned}\frac{1}{\beta V}\sum_{k',\sigma} &\left[G^{-1}_{0,\sigma'}(k'+q) - G^{-1}_{0,\sigma'}(k')\right] G_{\sigma'}(k')G_{\sigma'}(k'+q) \\ &\times \Gamma^{(4)}_{\mathrm{ph},\sigma\sigma\sigma'\sigma'}(k+q,k,k',k'+q) = -\Sigma_\sigma(k+q) + \Sigma_\sigma(k),\end{aligned} \tag{2.170}$$

Spin rotation

For a rotation of angle $\epsilon(x)$ about the $\mathbf{n}$ axis in spin space,

$$\begin{aligned}\psi(x) &\to e^{-\frac{i}{2}\epsilon(x)\sigma^{(n)}}\psi(x) = \psi(x) - \frac{i}{2}\epsilon(x)\sigma^{(n)}\psi(x) + \mathcal{O}(\epsilon^2), \\ \psi^\dagger(x) &\to \psi^\dagger(x)e^{\frac{i}{2}\epsilon(x)\sigma^{(n)}} = \psi^\dagger(x) + \frac{i}{2}\epsilon(x)\psi^\dagger(x)\sigma^{(n)} + \mathcal{O}(\epsilon^2),\end{aligned} \tag{2.171}$$

where $\psi(x) = (\psi_\uparrow(x),\psi_\downarrow(x))^T$, $\psi^\dagger(x) = (\psi^*_\uparrow(x),\psi^*_\downarrow(x))$ and $\sigma^{(n)} = \boldsymbol{\sigma}\cdot\mathbf{n}$. This gives $t^{(-)}_{\sigma\sigma'}(x,x') = -\frac{1}{2}\delta(x-x')\sigma^{(n)}_{\sigma\sigma'}$ and $t^{(+)}_{\sigma\sigma'}(x,x') = \frac{1}{2}\delta(x-x')\sigma^{(n)*}_{\sigma\sigma'}$, and in turn

$$\begin{aligned}\frac{1}{\beta V}\sum_{k',\sigma,\sigma'} &\left[-\sigma^{(n)}_{\sigma'\sigma}G^{-1}_{0,\sigma'}(k'+q) + \sigma^{(n)*}_{\sigma\sigma'}G^{-1}_{0,\sigma'}(k')\right] G_\sigma(k')G_{\sigma'}(k'+q) \\ &\times \Gamma^{(4)}_{\mathrm{ph},\sigma_2\sigma_1\sigma\sigma'}(k+q,k,k',k'+q) = \sigma^{(n)}_{\sigma_2\sigma_1}\Sigma_{\sigma_2}(k+q) - \sigma^{(n)*}_{\sigma_1\sigma_2}(k+q)\Sigma_{\sigma_1}(k).\end{aligned} \tag{2.172}$$

Considering $\mathbf{n}=\mathbf{z}$ and $\mathbf{n}=\mathbf{x}$, we deduce the Ward identities

$$\begin{aligned}\frac{1}{\beta V}\sum_{k',\sigma'} \sigma' &\left[G^{-1}_{0,\sigma'}(k'+q) - G^{-1}_{0,\sigma'}(k')\right] G_{\sigma'}(k')G_{\sigma'}(k'+q) \\ &\times \Gamma^{(4)}_{\mathrm{ph},\sigma\sigma\sigma'\sigma'}(k+q,k,k',k'+q) = -\sigma\left[\Sigma_\sigma(k+q) - \Sigma_\sigma(k)\right],\end{aligned} \tag{2.173}$$

and

$$\frac{1}{\beta V}\sum_{k',\sigma'}\left[G^{-1}_{0,\bar\sigma'}(k'+q)-G^{-1}_{0,\sigma'}(k')\right]G_{\sigma'}(k')G_{\bar\sigma'}(k'+q) \\ \times\Gamma^{(4)}_{\text{ph},\bar\sigma\sigma\sigma'\bar\sigma'}(k+q,k,k',k'+q)=-\Sigma_{\bar\sigma}(k+q)+\Sigma_\sigma(k). \tag{2.174}$$

For a spin-rotation invariant system, the particle-hole vertex can be written as

$$\Gamma^{(4)}_{\text{ph},\sigma_1\sigma_{1'}\sigma_2\sigma_{2'}}=\Gamma^{(4)}_{\text{ch}}\sigma^0_{\sigma_1\sigma_{1'}}\sigma^0_{\sigma_2\sigma_{2'}}+\Gamma^{(4)}_{\text{sp}}\boldsymbol{\sigma}_{\sigma_1\sigma_{1'}}\cdot\boldsymbol{\sigma}_{\sigma_2\sigma_{2'}}. \tag{2.175}$$

The three Ward identities (2.170,2.173,2.174) due to gauge and spin-rotation invariances then reduce to

$$\frac{2}{\beta V}\sum_{k'}\left[G^{-1}_0(k'+q)-G^{-1}_0(k')\right]G(k')G(k'+q) \\ \times\Gamma^{(4)}_{\text{ch/sp}}(k+q,k,k',k'+q)=-\Sigma(k+q)+\Sigma(k), \tag{2.176}$$

where the one-particle Green function and the self-energy are now spin independent.

Galilean transformation

Let us finally consider the "Galilean" transformation

$$\begin{aligned}\psi_\sigma(x)&\to\psi_\sigma(\mathbf{r}-\mathbf{R}(\tau),\tau)=\psi_\sigma(x)-\mathbf{R}(\tau)\cdot\boldsymbol{\nabla}\cdot\psi_\sigma(x)+\mathcal{O}(\mathbf{R}^2),\\ \psi^*_\sigma(x)&\to\psi^*_\sigma(\mathbf{r}-\mathbf{R}(\tau),\tau)=\psi^*_\sigma(x)-\mathbf{R}(\tau)\cdot\boldsymbol{\nabla}\cdot\psi^*_\sigma(x)+\mathcal{O}(\mathbf{R}^2),\end{aligned} \tag{2.177}$$

which amounts to viewing the system in a frame moving with a time-dependent velocity $\mathbf{v}(\tau)=id\mathbf{R}/d\tau$.[36] If the interaction S_{int} is local in time and translation invariant, it is invariant in this transformation. Equation (2.177) defines a linear transformation with $t^{(\pm)}_{\sigma\sigma'}(x,x')=i\delta_{\sigma,\sigma'}\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{\nabla}_{\mathbf{r}'}\delta(x-x')$, where we assume $\mathbf{R}(\tau)$ to be parallel to the unit vector $\hat{\boldsymbol{\epsilon}}$. The equation (2.145) then gives

$$\sum_{\sigma_1}\int dx_{1'}dx_{2'}dx_{3'}\Big[G^{-1}_{0,\sigma_1}(x_{1'},x_1)\boldsymbol{\nabla}_{\mathbf{r}_1}G_{\sigma_1}(x_1,x_{2'})G_{\sigma_1}(x_{3'},x_{1'}) \\ +G^{-1}_{0,\sigma_1}(x_1,x_{1'})G_{\sigma_1}(x_{1'},x_{2'})\boldsymbol{\nabla}_{\mathbf{r}_1}G_{\sigma_1}(x_{3'},x_1)\Big]\Gamma^{(4)}_{\text{ph},\sigma_2\sigma_2\sigma_1\sigma_1}(x_3,x_2,x_{2'},x_{3'}) \\ =-\boldsymbol{\nabla}_{\mathbf{r}_2}\delta(x_1-x_2)\Sigma_{\sigma_2}(x_3-x_1)-\boldsymbol{\nabla}_{\mathbf{r}_3}\delta(x_1-x_3)\Sigma_{\sigma_2}(x_1-x_2). \tag{2.178}$$

In Fourier space, we finally obtain the Ward identity

$$\frac{1}{\beta V}\sum_{k',\sigma'}\left[\mathbf{k}'G^{-1}_{0,\sigma'}(k'+q)-(\mathbf{k}'+\mathbf{q})G^{-1}_{0,\sigma'}(k')\right]G_{\sigma'}(k')G_{\sigma'}(k'+q) \\ \times\Gamma^{(4)}_{\text{ph},\sigma\sigma\sigma'\sigma'}(k+q,k,k',k'+q)=-\mathbf{k}\Sigma_\sigma(k+q)+(\mathbf{k}+\mathbf{q})\Sigma_\sigma(k). \tag{2.179}$$

[36] In real time ($\tau=it$), $\mathbf{v}(t)=d\mathbf{R}/dt$.

Guide to the bibliography

Symmetries in quantum mechanics are discussed in a number of books. Here I mention only those that were most useful when writing this chapter [1–5]. Sections 2.1 and 2.2 are mainly based on Ref. [4]. Symmetries in quantum field theory are discussed in Refs. [4, 5]. Ward identities were first obtained in quantum electrodynamics [6,7]. For condensed-matter systems, they are discussed in Refs. [8–11]. Goldstone theorem was first derived in Refs. [12–14]. To a large extent our discussion follows Ref. [5].

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