Chapter 8

Quantum magnetism *(last version: 20 August 2019)*

Contents

This chapter is devoted to quantum magnetism. Our aim is not to give an exhaustive introduction to the subject but to discuss a few key concepts and theoretical methods. After an introduction to the Heisenberg model, its classical dynamics and its mean-field solution, we discuss spinwaves in ferromagnets and antiferromagnets using both a random-phase approximation and Holstein-Primakoff bosons (Secs. 8.1-8.3). Then we introduce the spin-coherent-state functional integral and show that the low-energy behavior of an antiferromagnet is governed by a quantum nonlinear sigma model with a topological Berry phase term (Sec. 8.4). Finally we discuss a mean-field theory of the Heisenberg model based on a Schwinger-boson representation of the spin operator (Sec. 8.5).

8.1 The Heisenberg model

The Heisenberg model is defined by the Hamiltonian[1]

$$\hat{H} = -\frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'}\hat{\mathbf{S}}_{\mathbf{r}}\cdot\hat{\mathbf{S}}_{\mathbf{r}'}, \tag{8.1}$$

where $\{\mathbf{r}\}$ denotes the N sites of a d-dimensional hypercubic lattice and the lattice spacing is taken as the unit length. $\hat{\mathbf{S}}_{\mathbf{r}} = (\hat{S}^x_{\mathbf{r}}, \hat{S}^y_{\mathbf{r}}, \hat{S}^z_{\mathbf{r}})$ is a spin-S operator satisfying $\hat{\mathbf{S}}^2_{\mathbf{r}} = S(S+1)$ and the commutation relations

$$[\hat{S}^\nu_{\mathbf{r}}, \hat{S}^{\nu'}_{\mathbf{r}'}] = i\sum_{\nu''=x,y,z}\epsilon_{\nu\nu'\nu''}\delta_{\mathbf{r},\mathbf{r}'}\hat{S}^{\nu''}_{\mathbf{r}}. \tag{8.2}$$

In the following, we shall consider mainly the case where the spin-spin interaction $J_{\mathbf{r},\mathbf{r}'}$ equals J if $\mathbf{r}$ and $\mathbf{r}'$ are nearest neighbors and vanishes otherwise. The Fourier transforms of $J_{\mathbf{r},\mathbf{r}'}$ is then given by

$$J(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{r},\mathbf{r}'} e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}J_{\mathbf{r},\mathbf{r}'} = 2J\sum_{\mu=1}^{d}\cos q_\mu. \tag{8.3}$$

Solving the Heisenberg model is in general a very difficult task. The problem simplifies in the large-S limit where the spin operators $\hat{\mathbf{S}}_{\mathbf{r}}$ can be treated as classical spin variables of length $|\mathbf{S}_{\mathbf{r}}| = S$, since the commutator $[\hat{S}^\nu_{\mathbf{r}}, \hat{S}^{\nu'}_{\mathbf{r}'}] = \mathcal{O}(S)$ [Eq. (8.2)] can be neglected wrt the product $\hat{S}^\nu_{\mathbf{r}}\hat{S}^{\nu'}_{\mathbf{r}'}$, the latter being $\mathcal{O}(S^2)$. Finding the ground state of the system then amounts to minimizing the classical energy

$$E_{\rm cl} = -\frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'}\mathbf{S}_{\mathbf{r}}\cdot\mathbf{S}_{\mathbf{r}'} = -\frac{1}{2}\sum_{\mathbf{q}} J(\mathbf{q})\mathbf{S}_{-\mathbf{q}}\cdot\mathbf{S}_{\mathbf{q}} \tag{8.4}$$

with the constraint $|\mathbf{S}_{\mathbf{r}}| = S$. In the simple case we are considering [Eq. (8.3)], the solution is straightforward. For $J > 0$, the ground state is ferromagnetic with all spins pointing in the same direction, e.g. $\mathbf{S}_{\mathbf{r}} = S\mathbf{u}_z$. For $J < 0$, the ground state is antiferromagnetic with nearest-neighbor spins antiparallel, e.g. $\mathbf{S}_{\mathbf{r}} = S(-1)^{\mathbf{r}}\mathbf{u}_z$ (Néel state).[2]

More generally, for arbitrary lattice and interactions, the ground state is determined by the maximum of $J(\mathbf{q})$. For example, if $J(\mathbf{q})$ is maximum at $\pm\mathbf{q}_0$, one has

$$\mathbf{S}_{\mathbf{r}} = \frac{1}{\sqrt{N}}[\mathbf{S}_{\mathbf{q}_0}e^{i\mathbf{q}_0\cdot\mathbf{r}} + \mathbf{S}_{-\mathbf{q}_0}e^{-i\mathbf{q}_0\cdot\mathbf{r}}] \tag{8.5}$$

with $\mathbf{S}_{-\mathbf{q}_0} = \mathbf{S}^*_{\mathbf{q}_0}$ since $\mathbf{S}_{\mathbf{r}}$ is real, i.e.

$$\mathbf{S}_{\mathbf{r}} = S[\mathbf{a}\cos(\mathbf{q}_0\cdot\mathbf{r}+\theta) + \mathbf{b}\sin(\mathbf{q}_0\cdot\mathbf{r}+\theta)], \tag{8.6}$$

where $\mathbf{a}$ and $\mathbf{b}$ are perpendicular unit vectors (thus ensuring that the condition $|\mathbf{S}_{\mathbf{r}}| = S$ is fulfilled) and θ an arbitrary phase. Ferromagnetism and antiferromagnetism

[1] The Heisenberg model is the large-U limit of the half-filled Hubbard model; see Sec. 6.4.4.
[2] $(-1)^{\mathbf{r}} \equiv e^{i\mathbf{Q}\cdot\mathbf{r}}$ with $\mathbf{Q} = (\pi, \cdots, \pi)$.

correspond to $\mathbf{q}_0 = 0$ and $\mathbf{q}_0 = (\pi, \cdots, \pi)$, respectively. In other cases, equation (8.6) corresponds to spiral (or helicoidal) order.

The large-S limit provides us with a convenient starting point to study the effect of quantum fluctuations. Most of the discussion in this chapter, except the Schwinger-boson theory in section 8.5, is based on a $1/S$ expansion.

8.1.1 The ferromagnet

In the case of the ferromagnet, it turns out that the classical spin configuration with lowest energy also determines the quantum ground state

$$|0\rangle = \prod_{\mathbf{r}} |S\rangle_{\mathbf{r}}, \tag{8.7}$$

where $|S\rangle_{\mathbf{r}}$ denotes the eigenstate of $\hat{S}^z_{\mathbf{r}}$ with the highest possible eigenvalue $S^z = S$. By writing the Hamiltonian as

$$\hat{H} = -J \sum_{\langle \mathbf{r},\mathbf{r}'\rangle} \left(\hat{S}^z_{\mathbf{r}} \hat{S}^z_{\mathbf{r}'} + \frac{1}{2} \hat{S}^+_{\mathbf{r}} \hat{S}^-_{\mathbf{r}'} + \frac{1}{2} \hat{S}^-_{\mathbf{r}} \hat{S}^+_{\mathbf{r}'} \right) \tag{8.8}$$

(the sum is over all pairs of nearest neighbors) where $\hat{S}^\pm_{\mathbf{r}} = \hat{S}^x_{\mathbf{r}} \pm i \hat{S}^y_{\mathbf{r}}$, one immediately sees that $|0\rangle$ is an eigenstate of $\hat{H}$ with an eigenvalue given by the classical energy $E_{\rm cl} = -NJS^2 d$ since $\hat{S}^+_{\mathbf{r}} \hat{S}^-_{\mathbf{r}'}|0\rangle = \hat{S}^-_{\mathbf{r}} \hat{S}^+_{\mathbf{r}'}|0\rangle = 0$.[3] To show that $E_{\rm cl}$ is the lowest eigenvalue of $\hat{H}$, let us consider an eigenstate $|0'\rangle$ with eigenvalue $E'_0 = \langle 0'|\hat{H}|0'\rangle = -J\sum_{\langle \mathbf{r},\mathbf{r}'\rangle} \langle 0'|\hat{\mathbf{S}}_{\mathbf{r}} \cdot \hat{\mathbf{S}}_{\mathbf{r}'}|0'\rangle$. Now the largest value of $\langle 0'|\hat{\mathbf{S}}_{\mathbf{r}} \cdot \hat{\mathbf{S}}_{\mathbf{r}'}|0'\rangle$ is equal to the largest eigenvalue of $\hat{\mathbf{S}}_{\mathbf{r}} \cdot \hat{\mathbf{S}}_{\mathbf{r}'}$, that is S^2, so that $E_{0'} \geq -J\sum_{\langle \mathbf{r},\mathbf{r}'\rangle} S^2 = E_{\rm cl}$.[4]

It is also possible to find some of the low-lying excited states of the ferromagnetic Heisenberg model. Let us consider the normalized excited state

$$|\mathbf{r}\rangle = \frac{1}{\sqrt{2S}} \hat{S}^-_{\mathbf{r}} |0\rangle \tag{8.9}$$

obtained by decreasing S^z from S to $S-1$ for the spin at site $\mathbf{r}$. This state is not an eigenstate of the Hamiltonian since

$$(\hat{S}^-_{\mathbf{r}_1} \hat{S}^+_{\mathbf{r}_2} + \hat{S}^+_{\mathbf{r}_1} \hat{S}^-_{\mathbf{r}_2})|\mathbf{r}\rangle = 2S(\delta_{\mathbf{r},\mathbf{r}_2}|\mathbf{r}_1\rangle + \delta_{\mathbf{r},\mathbf{r}_1})|\mathbf{r}_2\rangle \qquad (\mathbf{r}_1 \neq \mathbf{r}_2). \tag{8.10}$$

But it is easy to show that the linear combination

$$|\mathbf{q}\rangle = \frac{1}{\sqrt{N}} \sum_{\mathbf{r}} e^{i\mathbf{q}\cdot\mathbf{r}} |\mathbf{r}\rangle, \qquad \hat{H}|\mathbf{q}\rangle = (E_{\rm cl} + \omega_{\mathbf{q}})|\mathbf{q}\rangle, \tag{8.11}$$

is an eigenstate with an excitation energy

$$\omega_{\mathbf{q}} = S[J(0) - J(\mathbf{q})] \tag{8.12}$$

[3] $\hat{S}^\pm |S^z\rangle = [(S \mp S^z)(S + 1 \pm S^z)]^{1/2} |S^z \pm 1\rangle$ implies that $\hat{S}^+|S\rangle = 0$.

[4] Write $\hat{\mathbf{S}}_{\mathbf{r}} \cdot \hat{\mathbf{S}}_{\mathbf{r}'} = \frac{1}{2}[(\hat{\mathbf{S}}_{\mathbf{r}} + \hat{\mathbf{S}}_{\mathbf{r}'})^2 - \hat{\mathbf{S}}^2_{\mathbf{r}} - \hat{\mathbf{S}}^2_{\mathbf{r}'}]$ and notice that $\hat{\mathbf{S}}^2_{\mathbf{r}} = S(S+1)$ and $(\hat{\mathbf{S}}_{\mathbf{r}} + \hat{\mathbf{S}}_{\mathbf{r}'})^2 = F(F+1)$ with $0 \leq F \leq 2S$. It follows that the largest and smallest eigenvalues of $\hat{\mathbf{S}}_{\mathbf{r}} \cdot \hat{\mathbf{S}}_{\mathbf{r}'}$ are S^2 and $-S(S+1)$, respectively.

which behaves as $JS\mathbf{q}^2$ in the long wavelength limit $|\mathbf{q}| \ll 1$. To give a physical interpretation of the state $|\mathbf{q}\rangle$, let us consider the transverse spin correlation function[5]

$$\langle \mathbf{q}|\hat{\mathbf{S}}^\perp_{\mathbf{r}} \cdot \hat{\mathbf{S}}^\perp_{\mathbf{r}'}|\mathbf{q}\rangle = \frac{1}{2}\langle \mathbf{q}|\hat{S}^-_{\mathbf{r}}\hat{S}^+_{\mathbf{r}'} + \hat{S}^+_{\mathbf{r}}\hat{S}^-_{\mathbf{r}'}|\mathbf{q}\rangle = \frac{2S}{N}\cos[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')] \qquad (\mathbf{r} \neq \mathbf{r}'), \tag{8.13}$$

where $\hat{\mathbf{S}}^\perp_{\mathbf{r}} = (\hat{S}^x_{\mathbf{r}}, \hat{S}^y_{\mathbf{r}})$ and we have used (8.10). Thus on average the orientations of the transverse (i.e. perpendicular to the magnetization) component of two spins separated by a distance $\mathbf{r}-\mathbf{r}'$ differ by an angle $\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')$. One can therefore see the state $|\mathbf{q}\rangle$ as a state containing a spinwave, or "magnon", of wave vector $\mathbf{q}$ and energy $\omega_{\mathbf{q}}$.

One can define creation and annihilation operators for magnons *via* $|\mathbf{q}\rangle = \hat{a}^\dagger_{\mathbf{q}}|0\rangle$,[6] i.e.

$$\hat{a}^\dagger_{\mathbf{q}} = \frac{1}{\sqrt{2S}}\hat{S}^-_{-\mathbf{q}}, \qquad \hat{a}_{\mathbf{q}} = \frac{1}{\sqrt{2S}}\hat{S}^+_{\mathbf{q}}. \tag{8.14}$$

These operators are not bosonic operators since the commutator

$$[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{q}'}] = \frac{1}{NS}\sum_{\mathbf{r}} e^{-i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{r}}\hat{S}^z_{\mathbf{r}} \tag{8.15}$$

does not reduce to $\delta_{\mathbf{q},\mathbf{q}'}$. However, given that $[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{q}'}]|0\rangle = \delta_{\mathbf{q},\mathbf{q}'}|0\rangle$, one can assume the magnons to obey bosonic statistics if their density is low enough, which is expected to be the case at low temperatures. Furthermore, when the magnon density is small, one can neglect the interactions between magnons. The mean number of magnons with momentum $\mathbf{q}$ is then given by the Bose-Einstein statistics, $\langle \hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}\rangle = n_B(\omega_{\mathbf{q}})$, and it is straightforward to compute thermodynamic quantities. For example, at finite temperature the magnetization per site is given by

$$\langle \hat{S}^z_{\mathbf{r}}\rangle_T = \langle \hat{S}^z_{\mathbf{r}}\rangle_{T=0} - \frac{1}{N}\sum_{\mathbf{q}} n_B(\omega_{\mathbf{q}}) = S - \int_{\mathbf{q}} \frac{1}{e^{\beta\omega_{\mathbf{q}}} - 1}. \tag{8.16}$$

At small temperature, $T \ll JS$, the momentum integral in (8.16) is dominated by small $\mathbf{q}$'s and one can approximate $\omega_{\mathbf{q}}$ by $JS\mathbf{q}^2$. In three dimensions, this leads to the Bloch $T^{3/2}$ law: $S - \langle \hat{S}^z_{\mathbf{r}}\rangle_T \propto T^{3/2}$. In two dimensions, the integral in (8.16) is infrared divergent, which suggests the absence of long-range order at finite temperature, in agreement with the Mermin-Wagner theorem (Sec. 3.6.4). A similar calculation shows that in three dimensions the energy varies as $T^{5/2}$ and the specific heat as $T^{3/2}$.

8.1.2 The antiferromagnet

In the case of the antiferromagnet, the classical spin configuration with minimum energy yields the Néel state

$$|0\rangle = \prod_{\mathbf{r}} |(-1)^{\mathbf{r}} S\rangle_{\mathbf{r}}, \tag{8.17}$$

[5] Note that $\langle \mathbf{q}|\hat{\mathbf{S}}^\perp_{\mathbf{r}}|\mathbf{q}\rangle = 0$, which is the reason why we consider the correlation function (8.13).

[6] Fourier transformed operators are defined by $\hat{a}_{\mathbf{q}} = \frac{1}{\sqrt{N}}\sum_{\mathbf{r}} e^{-i\mathbf{q}\cdot\mathbf{r}}\hat{a}_{\mathbf{r}}$, $\hat{a}^\dagger_{\mathbf{q}} = \frac{1}{\sqrt{N}}\sum_{\mathbf{r}} e^{i\mathbf{q}\cdot\mathbf{r}}\hat{a}^\dagger_{\mathbf{r}}$ and $\hat{\mathbf{S}}_{\mathbf{q}} = \frac{1}{\sqrt{N}}\sum_{\mathbf{r}} e^{-i\mathbf{q}\cdot\mathbf{r}}\hat{\mathbf{S}}_{\mathbf{r}}$.

which is not the ground state of the quantum Hamiltonian. Contrary to the ferromagnet, the operator $\hat{S}^+_{\mathbf{r}}\hat{S}^-_{\mathbf{r}'}$ acting on nearest-neighbor spins does not always vanish since $\hat{S}^+_{\mathbf{r}}\hat{S}^-_{\mathbf{r}'}|-S\rangle_{\mathbf{r}}|S\rangle_{\mathbf{r}'} \propto |-S+1\rangle_{\mathbf{r}}|S-1\rangle_{\mathbf{r}'}$. This process can be seen as a manifestation of quantum fluctuations. Is there long-range order in the antiferromagnetic Heisenberg model?[7] What are the low-lying excitations? We shall try to answer these questions in the following sections.

The only case where the ground state of the Heisenberg model is known exactly is the one-dimensional spin-$\frac{1}{2}$ chain: spin-rotation is not broken in the ground state but the $q=\pi$ spin-spin correlation function decays algebraically (not exponentially). The elementary excitations form a particle-hole continuum $\omega_q = \epsilon_{q+k} - \epsilon_k$, obtained from fundamental excitations with dispersion law $\epsilon_k = \frac{\pi}{2}|J\sin k|$ which are usually called spinons.[8] More generally, for half-integer spin chains, there exists excited states whose energy vanishes when N is even and $N\to\infty$ (Lieb, Schultz and Mattis theorem [22]). We shall see that integer spin chains are fundamentally different and exhibit a gapped spectrum. In higher dimensions, whether S is integer or half-integer does not play an important role (Sec. 8.4.3).

8.2 Random-phase approximation

A first insight into the dynamics of the (anti)ferromagnet can be obtained from a purely classical analysis or the random-phase approximation. Of course, for the antiferromagnet, these approaches make sense only if quantum fluctuations do not suppress long-range order; we shall see that this is indeed the case for a hypercubic lattice in dimension $d \geq 2$.

8.2.1 Classical dynamics

We write the classical energy (8.4) as

$$E_{\rm cl} = -\frac{1}{2}\sum_{\mathbf{r}} \mathbf{h}_{\mathbf{r}}\cdot\boldsymbol{\mu}_{\mathbf{r}}, \tag{8.18}$$

where $\mathbf{h}_{\mathbf{r}} = -\frac{1}{g\mu_B}\sum_{\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'}\mathbf{S}_{\mathbf{r}'}$ is the effective magnetic field acting on the magnetic moment $\boldsymbol{\mu}_{\mathbf{r}} = -g\mu_B\mathbf{S}_{\mathbf{r}}$. From classical mechanics, the rate of change of the angular momentum $\mathbf{S}_{\mathbf{r}}$ is given by $\boldsymbol{\mu}_{\mathbf{r}}\times\mathbf{h}_{\mathbf{r}}$, i.e.

$$\frac{d}{dt}\mathbf{S}_{\mathbf{r}} = \boldsymbol{\mu}_{\mathbf{r}}\times\mathbf{h}_{\mathbf{r}} = \sum_{\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'}\mathbf{S}_{\mathbf{r}}\times\mathbf{S}_{\mathbf{r}'}. \tag{8.19}$$

Note that a factor 1/2 is introduced in (8.18) to avoid double counting of the spin-spin interactions. Assuming small deviations from the ground state, we linearize this equation about $\mathbf{S}_{\mathbf{r}} = S\mathbf{u}_z$ (ferromagnet) or $\mathbf{S}_{\mathbf{r}} = (-1)^{\mathbf{r}}S\mathbf{u}_z$ (antiferromagnet). For

[7] For a discussion about the meaning of spontaneously broken symmetry in antiferromagnets, see the end of Sec. 3.6.2.

[8] The spinon is not a spinwave, although it has a similar dispersion (see Sec. 8.3.2), since it does not correspond to small fluctuations about a ground state with broken symmetry.

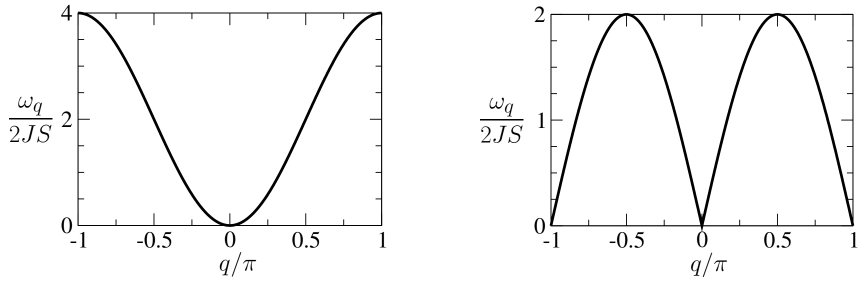

Figure 8.1: Spinwave dispersion ω_q in ferromagnets (left) and antiferromagnets (right). The dispersion is shown for $d = 1$.

the ferromagnet, one obtains

$$\begin{aligned}\frac{d}{dt} S^x_{\mathbf{q}} &= S[J(0) - J(\mathbf{q})] S^y_{\mathbf{q}}, \\ \frac{d}{dt} S^y_{\mathbf{q}} &= S[J(\mathbf{q}) - J(0)] S^x_{\mathbf{q}}.\end{aligned} \tag{8.20}$$

Looking for plane wave solutions $S^{x,y}_{\mathbf{q}} \sim e^{-i\omega t}$, one recovers the magnon excitation energy (8.12) vanishing quadratically for $\mathbf{q} \to 0$ (Fig. 8.1).

In the case of the antiferromagnet,

$$\begin{aligned}\frac{d}{dt} S^x_{\mathbf{q}} &= S[J(\mathbf{Q}) - J(\mathbf{q}+\mathbf{Q})] S^y_{\mathbf{q}+\mathbf{Q}}, \\ \frac{d}{dt} S^y_{\mathbf{q}+\mathbf{Q}} &= -S[J(\mathbf{Q}) - J(\mathbf{q})] S^x_{\mathbf{q}},\end{aligned} \tag{8.21}$$

where $\mathbf{Q} = (\pi, \cdots, \pi)$ and $J(\mathbf{q}+\mathbf{Q}) = -J(\mathbf{q})$, which gives

$$\omega_{\mathbf{q}} = S\sqrt{J(0)^2 - J(\mathbf{q})^2}. \tag{8.22}$$

Unlike ferromagnetic spinwaves, there are two modes, $\mathbf{q} = 0$ and $\mathbf{q} = \mathbf{Q}$, where the energy vanishes. Furthermore, the energy vanishes linearly with momentum, i.e. $\omega_{\mathbf{q}} = c|\mathbf{q}|$ for $\mathbf{q} \to 0$ or $\omega_{\mathbf{q}} = c|\mathbf{q} - \mathbf{Q}|$ for $\mathbf{q} \to \mathbf{Q}$, with a velocity $c = 2\sqrt{d}|J|S$ (Fig. 8.1).

8.2.2 Spin susceptibilities in the random-phase approximation

The spinwave modes can also be obtained from a random-phase approximation (RPA). Let us start with a static mean-field approximation obtained by decoupling the spin-spin interaction,[9]

$$\begin{aligned}\hat{H}_{\text{MF}} &= -\frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} \big(\langle \hat{\mathbf{S}}_{\mathbf{r}} \rangle \cdot \hat{\mathbf{S}}_{\mathbf{r}'} + \hat{\mathbf{S}}_{\mathbf{r}} \cdot \langle \hat{\mathbf{S}}_{\mathbf{r}'} \rangle - \langle \hat{\mathbf{S}}_{\mathbf{r}} \rangle \cdot \langle \hat{\mathbf{S}}_{\mathbf{r}'} \rangle \big) \\ &= -\sum_{\mathbf{r}} \mathbf{h}_{\mathbf{r}} \cdot \hat{\mathbf{S}}_{\mathbf{r}} + \text{const},\end{aligned} \tag{8.23}$$

[9]Eq. (8.23) is obtained by linearizing the Hamiltonian in the fluctuation operator $\delta\hat{\mathbf{S}}_{\mathbf{r}} = \hat{\mathbf{S}}_{\mathbf{r}} - \langle \hat{\mathbf{S}}_{\mathbf{r}} \rangle$.

where

$$\mathbf{h}_\mathbf{r} = \sum_{\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} \langle \hat{\mathbf{S}}_{\mathbf{r}'} \rangle = mJ(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} \mathbf{u}_z \tag{8.24}$$

is the effective field acting on the spin located at site $\mathbf{r}$. The last expression in (8.24) is obtained by assuming $\langle \hat{\mathbf{S}}_\mathbf{r} \rangle = me^{i\mathbf{q}\cdot\mathbf{r}}\mathbf{u}_z$ with $\mathbf{q} = 0$ (ferromagnet) or $\mathbf{q} = \mathbf{Q}$ (antiferromagnet). The mean-field Hamiltonian (8.23) describes a set of decoupled spins and can be easily solved. The magnetization m is obtained from the self-consistent equation

$$\langle \hat{S}^z_\mathbf{r} \rangle = \frac{1}{Z} \mathrm{Tr} \left(e^{-\beta \hat{H}_{\mathrm{MF}}} \hat{S}^z_\mathbf{r} \right), \tag{8.25}$$

which gives (see Appendix 8.A)

$$m = SB_S(\beta hS), \tag{8.26}$$

where $h = mJ(\mathbf{q})$ and

$$B_S(x) = \frac{2S+1}{2S} \operatorname{cotanh} \left(\frac{2S+1}{2S} x \right) - \frac{1}{2S} \operatorname{cotanh} \left(\frac{x}{2S} \right) \tag{8.27}$$

is the Brillouin function. At zero temperature, $m = SB_S(\infty) = S$, which is the expected result in a mean-field theory neglecting quantum fluctuations. For an infinitesimal magnetization, using $B_S(x) = x(S+1)/3S + \mathcal{O}(x^3)$, one can linearize equation (8.26) to obtain the transition temperature

$$T_c = \frac{S(S+1)}{3} J(\mathbf{q}) = \frac{2}{3} |J| dS(S+1). \tag{8.28}$$

Note that T_c takes the same expression for the ferromagnet ($J > 0$) and the antiferromagnet ($J < 0$).

Let us now consider the system in the presence of a time-dependent external field $\mathbf{h}^{\mathrm{ext}}_\mathbf{r}(t) = \mathbf{h}^{\mathrm{ext}} e^{i(\mathbf{q}\cdot\mathbf{r} - \omega t)} + \text{c.c.}$ (with $\mathbf{q}$ now arbitrary),

$$\hat{H} = -\frac{1}{2} \sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} \hat{\mathbf{S}}_\mathbf{r} \cdot \hat{\mathbf{S}}_{\mathbf{r}'} - \sum_\mathbf{r} \mathbf{h}^{\mathrm{ext}}_\mathbf{r}(t) \cdot \hat{\mathbf{S}}_\mathbf{r}. \tag{8.29}$$

The RPA amounts to linearizing the Hamiltonian wrt the fluctuations $\delta \hat{\mathbf{S}}_\mathbf{r}(t) = \hat{\mathbf{S}}_\mathbf{r} - \langle \hat{\mathbf{S}}_\mathbf{r} \rangle(t)$ of the spin operator,

$$\hat{H}_{\mathrm{RPA}} = -\sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} \langle \hat{\mathbf{S}}_\mathbf{r} \rangle(t) \cdot \hat{\mathbf{S}}_{\mathbf{r}'} - \sum_\mathbf{r} \mathbf{h}^{\mathrm{ext}}_\mathbf{r}(t) \cdot \hat{\mathbf{S}}_\mathbf{r}. \tag{8.30}$$

Since $\langle \hat{\mathbf{S}}_\mathbf{r} \rangle(t) = \langle \hat{\mathbf{S}}_\mathbf{r} \rangle + \delta \langle \hat{\mathbf{S}}_\mathbf{r} \rangle(t)$, with $\langle \hat{\mathbf{S}}_\mathbf{r} \rangle$ the mean value of the spin operator when $\mathbf{h}^{\mathrm{ext}} = 0$, we obtain the RPA Hamiltonian

$$\hat{H}_{\mathrm{RPA}} = \hat{H}_{\mathrm{MF}} - \sum_\mathbf{r} \mathbf{h}_\mathbf{r}(t) \cdot \hat{\mathbf{S}}_\mathbf{r}, \tag{8.31}$$

where

$$\mathbf{h}_\mathbf{r}(t) = \mathbf{h}^{\mathrm{ext}}_\mathbf{r}(t) + \sum_{\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} \delta \langle \hat{\mathbf{S}}_{\mathbf{r}'} \rangle(t) \tag{8.32}$$

is the time-dependent effective field acting on the spin located at site $\mathbf{r}$. Thus the RPA Hamiltonian boils down to a system of decoupled spins subject to a time-dependent field. The linear response to the field is given by the retarded spin-spin correlation function χ_0^R obtained from the mean-field Hamiltonian $\hat{H}_{\rm MF}$,[10]

$$\delta\langle \hat{S}^\nu_{\mathbf{r}}\rangle(t) = \int_{-\infty}^t dt' \sum_{\mathbf{r}',\nu'} \chi_0^{R\,\nu\nu'}(\mathbf{r},\mathbf{r}';t-t') h^{\nu'}_{\mathbf{r}'}(t'). \tag{8.33}$$

In matrix form, we can rewrite this equation as

$$\delta\langle \hat{S}\rangle = \chi_0^R h = \chi_0^R (J\delta\langle \hat{S}\rangle + h^{\rm ext}) \tag{8.34}$$

and obtain the retarded RPA susceptibility χ^R,

$$\delta\langle \hat{S}\rangle = (1-\chi_0^R J)^{-1}\chi_0^R h^{\rm ext} \equiv \chi^R h^{\rm ext}. \tag{8.35}$$

The spinwave spectrum is obtained from the divergence of the spin susceptibility, i.e. $\det\chi^{R-1} = \det(\chi_0^{R-1} - J) = 0$. The $T=0$ susceptibility χ_0^R is obtained from the Matsubara correlation function χ_0 calculated in Appendix 8.A [Eqs. (8.194,8.198)],

$$\begin{aligned} \chi_0^{R\,xx}(\mathbf{q},\mathbf{q},\omega) = \chi_0^{R\,yy}(\mathbf{q},\mathbf{q},\omega) &= \frac{Sh}{h^2-\omega^2}, \\ \chi_0^{R\,xy}(\mathbf{q},\mathbf{q}+\mathbf{Q},\omega) = -\chi_0^{R\,yx}(\mathbf{q},\mathbf{q}+\mathbf{Q},\omega) &= \frac{i\omega S}{h^2-\omega^2}, \end{aligned} \tag{8.36}$$

where $\mathbf{Q}=0$ (ferromagnet) or $\mathbf{Q}=(\pi,\cdots,\pi)$ (antiferromagnet), and $h = SJ(\mathbf{Q})$. The zz component $\chi_0^{R\,zz}$ of the spin susceptibility does not couple to the other components and is not considered here since we are interested only in the transverse (spinwave) excitations. Inverting the transverse part of χ_0^R, one finds that the spinwave spectrum is determined by

$$\det \frac{1}{S} \begin{pmatrix} S[J(\mathbf{Q})-J(\mathbf{q})] & -i\omega \\ i\omega & S[J(\mathbf{Q})-J(\mathbf{q}+\mathbf{Q})] \end{pmatrix} = 0, \tag{8.37}$$

which reproduces the previous result for the ferromagnet [Eq. (8.12)] and agrees with the classical analysis for the antiferromagnet [Eq. (8.22)].

8.3 Spin-waves: Holtstein-Primakoff bosons

In the large-S limit, spin operators become commuting classical variables and the ground state coincides with the classical ground state. It is possible to take into account quantum fluctuations in a $1/S$ expansion. A convenient way to do so is to write the spin operator in terms of Holstein-Primakoff bosons. The large-S expansion justifies the spinwave results obtained previously from the classical dynamics or the RPA.

[10] $\chi_0(\mathbf{r},\mathbf{r}';t-t') \propto \delta_{\mathbf{r},\mathbf{r}'}$ is obviously local but depends on $\mathbf{r}=\mathbf{r}'$ if $\langle \hat{\mathbf{S}}_{\mathbf{r}}\rangle$ is not uniform.

8.3.1 The ferromagnet

Let us consider the ferromagnetic ground state $|0\rangle$ with all spins pointing in the $\mathbf{u}_z$ direction. We parameterize deviations from the ground state by means of boson operators $\hat a_{\mathbf r}$ and $\hat a^\dagger_{\mathbf r}$,

$$\begin{aligned} \hat S^z_{\mathbf r} &= S - \hat n_{\mathbf r}, \\ \hat S^+_{\mathbf r} &= \sqrt{2S} f(\hat n_{\mathbf r})\hat a_{\mathbf r}, \qquad \hat n_{\mathbf r} = \hat a^\dagger_{\mathbf r}\hat a_{\mathbf r}, \qquad f(\hat n_{\mathbf r}) = \left(1 - \frac{\hat n_{\mathbf r}}{2S}\right)^{1/2}. \\ \hat S^-_{\mathbf r} &= \sqrt{2S}\hat a^\dagger_{\mathbf r} f(\hat n_{\mathbf r}), \end{aligned} \tag{8.38}$$

It is straightforward to show that the boson commutation relations $[\hat a_{\mathbf r}, \hat a^\dagger_{\mathbf r'}] = \delta_{\mathbf r,\mathbf r'}$, $[\hat a_{\mathbf r}, \hat a_{\mathbf r'}] = [\hat a^\dagger_{\mathbf r}, \hat a^\dagger_{\mathbf r'}] = 0$ imply $\hat S^2_{\mathbf r} = S(S+1)$ and the usual spin commutation relations,

$$\begin{aligned} [\hat S^+_{\mathbf r}, \hat S^-_{\mathbf r}] &= 2S[f(\hat n_{\mathbf r})\hat a_{\mathbf r}, \hat a^\dagger_{\mathbf r} f(\hat n_{\mathbf r})] = 2(S - \hat n_{\mathbf r}) = 2\hat S^z_{\mathbf r}, \\ [\hat S^+_{\mathbf r}, \hat S^z_{\mathbf r}] &= \sqrt{2S}[f(\hat n_{\mathbf r})\hat a_{\mathbf r}, S - \hat n_{\mathbf r}] = -\sqrt{2S} f(\hat n_{\mathbf r})\hat a_{\mathbf r} = -\hat S^+_{\mathbf r}, \\ [\hat S^-_{\mathbf r}, \hat S^z_{\mathbf r}] &= \sqrt{2S}[\hat a^\dagger_{\mathbf r} f(\hat n_{\mathbf r}), S - \hat n_{\mathbf r}] = \sqrt{2S}\hat a^\dagger_{\mathbf r} f(\hat n_{\mathbf r}) = \hat S^-_{\mathbf r}, \end{aligned} \tag{8.39}$$

i.e. $\hat{\mathbf S}_{\mathbf r} \times \hat{\mathbf S}_{\mathbf r} = i\hat{\mathbf S}_{\mathbf r}$. Since $S^z \in [-S, S]$, the physical Hilbert space for the bosons corresponds to $n_{\mathbf r} = \langle \hat a^\dagger_{\mathbf r}\hat a_{\mathbf r}\rangle \le 2S$. Given that $f(n = 2S) = 0$ the spin operators, expressed in terms of the boson operators [Eqs. (8.38)], do not couple the physical space $n_{\mathbf r} \le 2S$ to the nonphysical one $n_{\mathbf r} > 2S$.

As a function of the boson operators, the Heisenberg Hamiltonian (8.8) reads

$$\hat H = -\frac12 \sum_{\mathbf r,\mathbf r'} J_{\mathbf r,\mathbf r'}\left[(S - \hat n_{\mathbf r})(S - \hat n_{\mathbf r'}) + S f(\hat n_{\mathbf r})\hat a_{\mathbf r}\hat a^\dagger_{\mathbf r'} f(\hat n_{\mathbf r'}) + S\hat a^\dagger_{\mathbf r} f(\hat n_{\mathbf r}) f(\hat n_{\mathbf r'})\hat a_{\mathbf r'}\right]. \tag{8.40}$$

We can now consider the large-S limit and expand $\hat H$ in $1/S$. However, if we truncate the expansion (i.e. the function $f(n)$) to a finite order, then the spin operators $\hat S^\pm_{\mathbf r}$ will couple the physical space $n_{\mathbf r} \le 2S$ to the nonphysical one $n_{\mathbf r} > 2S$. For the expansion to make sense, we must therefore require $n_{\mathbf r} \ll 2S$, which is indeed the case since the ground state is close to the classical state when $S \gg 1$.

To leading (nontrivial) order, using $f(\hat n_{\mathbf r}) = 1 + \mathcal O(1/S)$, one finds

$$\begin{aligned} \hat H &= -\frac12 \sum_{\mathbf r,\mathbf r'} J_{\mathbf r,\mathbf r'}\left[S^2 - S(\hat n_{\mathbf r} + \hat n_{\mathbf r'}) + S(\hat a_{\mathbf r}\hat a^\dagger_{\mathbf r'} + \hat a^\dagger_{\mathbf r}\hat a_{\mathbf r'})\right] \\ &= E_{\rm cl} + \sum_{\mathbf q} \omega_{\mathbf q}\hat a^\dagger_{\mathbf q}\hat a_{\mathbf q}, \end{aligned} \tag{8.41}$$

where

$$\omega_{\mathbf q} = S[J(0) - J(\mathbf q)] \equiv 2dJS(1 - \gamma_{\mathbf q}), \qquad \gamma_{\mathbf q} = \frac{J(\mathbf q)}{J(0)} = \frac1d \sum_{\mu=1}^d \cos q_\mu \tag{8.42}$$

and $E_{\rm cl} = -\frac12 J(0)NS^2$ is the ground state energy. The Hamiltonian (8.41) describes noninteracting bosons with a dispersion which becomes quadratic in the long wavelength limit: $\omega_{\mathbf q} \simeq SJ\mathbf q^2$ for $|\mathbf q| \ll 1$. Since equation (8.41) was obtained using

$f(\hat{n}_{\mathbf{r}}) = 1$, we have $\hat{S}^-_{\mathbf{q}} = \sqrt{2S}\hat{a}^\dagger_{-\mathbf{q}}$ to leading order of the $1/S$ expansion, and we recover the fact that the magnon creation operator is $\hat{S}^-_{-\mathbf{q}}/\sqrt{2S}$ [Eq. (8.14)]. The ground state is the vacuum of bosons, $\langle \hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}\rangle = 0$, which corresponds to the classical spin configuration $\langle \hat{S}^z_{\mathbf{r}}\rangle = S - \langle \hat{a}^\dagger_{\mathbf{r}}\hat{a}_{\mathbf{r}}\rangle = S$.

The $\mathcal{O}(1/S^0)$ term, omitted in equation (8.41),

$$\hat{H}_{\rm int} = -\frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'}\Big(\hat{n}_{\mathbf{r}}\hat{n}_{\mathbf{r}'} - \frac{1}{2}\hat{n}_{\mathbf{r}}\hat{a}_{\mathbf{r}'}\hat{a}^\dagger_{\mathbf{r}'} - \frac{1}{2}\hat{a}_{\mathbf{r}}\hat{a}^\dagger_{\mathbf{r}'}\hat{n}_{\mathbf{r}'}\Big), \tag{8.43}$$

is quartic in the boson operators and describes interactions between magnons.

8.3.2 The antiferromagnet

The classical antiferromagnetic Néel state corresponds to $S^z_{\mathbf{r}} = S$ on sublattice A and $S^z_{\mathbf{r}} = -S$ on sublattice B.[11] It is convenient to define the Holstein-Primakoff bosons as follows:

$$\left\{\begin{array}{l} \hat{S}^z_{\mathbf{r}} = S - \hat{n}_{\mathbf{r}} \\ \hat{S}^+_{\mathbf{r}} = \sqrt{2S} f(\hat{n}_{\mathbf{r}})\hat{a}_{\mathbf{r}} \\ \hat{S}^-_{\mathbf{r}} = \sqrt{2S}\hat{a}^\dagger_{\mathbf{r}} f(\hat{n}_{\mathbf{r}}) \end{array}\right. \quad (\mathbf{r}\in {\rm A}), \qquad \left\{\begin{array}{l} \hat{S}^z_{\mathbf{r}} = -S + \hat{n}_{\mathbf{r}} \\ \hat{S}^+_{\mathbf{r}} = \sqrt{2S}\hat{a}^\dagger_{\mathbf{r}} f(\hat{n}_{\mathbf{r}}) \\ \hat{S}^-_{\mathbf{r}} = \sqrt{2S} f(\hat{n}_{\mathbf{r}})\hat{a}_{\mathbf{r}} \end{array}\right. \quad (\mathbf{r}\in {\rm B}), \tag{8.44}$$

so that the (classical) Néel state corresponds to the vacuum of bosons. The function $f(\hat{n})$ is defined as before [Eq. (8.38)] and one easily verifies that the spin commutation relations are satisfied on both sublattices.

To leading (nontrivial) order in the $1/S$ expansion,

$$\begin{aligned} \hat{H} &= E_{\rm cl} + \frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'} SJ_{\mathbf{r},\mathbf{r}'}\big[(-1)^{\mathbf{r}+\mathbf{r}'}(\hat{n}_{\mathbf{r}} + \hat{n}_{\mathbf{r}'}) - (\hat{a}_{\mathbf{r}}\hat{a}_{\mathbf{r}'} + {\rm h.c.})\big] \\ &= E_{\rm cl} - \frac{1}{2}\sum_{\mathbf{q}} S\big[2J(0)\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}} + J(\mathbf{q})(\hat{a}_{\mathbf{q}}\hat{a}_{-\mathbf{q}} + {\rm h.c.})\big], \end{aligned} \tag{8.45}$$

where $E_{\rm cl} = -\frac{1}{2}NS^2J(\mathbf{Q})$ is the energy of the classical Néel state and $\mathbf{Q} = (\pi,\cdots,\pi)$.$\hat{H}$ can be diagonalized by the following Bogoliubov transformation,

$$\begin{aligned} \hat{\alpha}^\dagger_{\mathbf{q}} &= u_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}} - v_{\mathbf{q}}\hat{a}_{-\mathbf{q}}, \\ \hat{\alpha}_{-\mathbf{q}} &= -v_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}} + u_{\mathbf{q}}\hat{a}_{-\mathbf{q}}, \end{aligned} \tag{8.46}$$

where $\hat{\alpha}_{\mathbf{q}}, \hat{\alpha}^\dagger_{\mathbf{q}}$ are bosonic operators. We assume that $u_{\mathbf{q}}$ and $v_{\mathbf{q}}$ are real and satisfy $u_{\mathbf{q}} = u_{-\mathbf{q}}$, $v_{\mathbf{q}} = v_{-\mathbf{q}}$. The condition $u^2_{\mathbf{q}} - v^2_{\mathbf{q}} = 1$ ensures that the transformation (8.46) is unitary: $[\hat{\alpha}_{\mathbf{q}}, \hat{\alpha}^\dagger_{\mathbf{q}'}] = \delta_{\mathbf{q},\mathbf{q}'}$ and $[\hat{\alpha}_{\mathbf{q}}, \hat{\alpha}_{\mathbf{q}'}] = [\hat{\alpha}^\dagger_{\mathbf{q}}, \hat{\alpha}^\dagger_{\mathbf{q}'}] = 0$. The Hamiltonian

$$\hat{H} = E_{\rm cl} - \frac{1}{2}J(0)S\sum_{\mathbf{q}}(\hat{\alpha}^\dagger_{\mathbf{q}}, \hat{\alpha}_{-\mathbf{q}})\begin{pmatrix} A_{\mathbf{q}} & B_{\mathbf{q}} \\ B_{\mathbf{q}} & A_{\mathbf{q}} \end{pmatrix}\begin{pmatrix} \hat{\alpha}_{\mathbf{q}} \\ \hat{\alpha}^\dagger_{-\mathbf{q}} \end{pmatrix}, \tag{8.47}$$

[11] Let us recall that we consider a d-dimensional hypercubic lattice, which can be divided into two sublattices A and B, with all sites in A having their nearest-neighbors in B and *vice versa*.

where

$$A_{\mathbf{q}} = u_{\mathbf{q}}^2 + v_{\mathbf{q}}^2 + 2\gamma_{\mathbf{q}} u_{\mathbf{q}} v_{\mathbf{q}}, \qquad B_{\mathbf{q}} = 2u_{\mathbf{q}} v_{\mathbf{q}} + \gamma_{\mathbf{q}}(u_{\mathbf{q}}^2 + v_{\mathbf{q}}^2), \tag{8.48}$$

becomes diagonal if we choose $u_{\mathbf{q}} = \cosh\theta_{\mathbf{q}}$ and $v_{\mathbf{q}} = \sinh\theta_{\mathbf{q}}$ with $\tanh(2\theta_{\mathbf{q}}) = -\gamma_{\mathbf{q}}$:

$$\hat{H} = E_{\rm cl} - \frac{1}{2} J(0) S \sum_{\mathbf{q}} (1-\gamma_{\mathbf{q}}^2)^{1/2} + \sum_{\mathbf{q}} \omega_{\mathbf{q}} \hat{\alpha}_{\mathbf{q}}^\dagger \hat{\alpha}_{\mathbf{q}}, \tag{8.49}$$

We recover the dispersion $\omega_{\mathbf{q}} = |J(0)|S(1-\gamma_{\mathbf{q}}^2)^{1/2}$ of the antiferromagnetic spinwave obtained from the classical dynamics and the RPA (Sec. 8.2). The ground state is defined by the vacuum of bosons $\hat{\alpha}, \hat{\alpha}^\dagger$ but differs from the vacuum of the Holstein-Primakoff bosons: $\langle \hat{\alpha}_{\mathbf{q}}^\dagger \hat{\alpha}_{\mathbf{q}} \rangle = 0$ but $\langle \hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{q}} \rangle \neq 0$. Because of the quantum fluctuations of the Holstein-Primakoff bosons, the ground-state energy is given by

$$E_0 = E_{\rm cl} - \frac{1}{2} J(0) S \sum_{\mathbf{q}} (1-\gamma_{\mathbf{q}}^2)^{1/2} \tag{8.50}$$

to order $\mathcal{O}(S)$. The staggered magnetization $(-1)^{\mathbf{r}} \langle \hat{S}_{\mathbf{r}}^z \rangle$, which is equal to S in the classical Néel state, is reduced by quantum fluctuations,

$$\begin{aligned} (-1)^{\mathbf{r}} \langle \hat{S}_{\mathbf{r}}^z \rangle &= S - \langle \hat{a}_{\mathbf{r}}^\dagger \hat{a}_{\mathbf{r}} \rangle \\ &= S - \int_{\mathbf{q}} \langle u_{\mathbf{q}}^2 \hat{\alpha}_{\mathbf{q}}^\dagger \hat{\alpha}_{\mathbf{q}} + v_{\mathbf{q}}^2 \hat{\alpha}_{\mathbf{q}} \hat{\alpha}_{\mathbf{q}}^\dagger + u_{\mathbf{q}} v_{\mathbf{q}} (\hat{\alpha}_{\mathbf{q}}^\dagger \hat{\alpha}_{-\mathbf{q}}^\dagger + \text{h.c.}) \rangle. \end{aligned} \tag{8.51}$$

Using $\langle \hat{\alpha}_{\mathbf{q}}^\dagger \hat{\alpha}_{-\mathbf{q}}^\dagger \rangle = \langle \hat{\alpha}_{\mathbf{q}} \hat{\alpha}_{-\mathbf{q}} \rangle = 0$ and $\langle \hat{\alpha}_{\mathbf{q}}^\dagger \hat{\alpha}_{\mathbf{q}} \rangle = n_B(\omega_{\mathbf{q}})$, one finally obtains

$$(-1)^{\mathbf{r}} \langle \hat{S}_{\mathbf{r}}^z \rangle = S - \frac{1}{2} \int_{\mathbf{q}} \left[-1 + \frac{2n_B(\omega_{\mathbf{q}}) + 1}{(1-\gamma_{\mathbf{q}}^2)^{1/2}} \right]. \tag{8.52}$$

We discuss below the consequences of this equation in one, two and three dimensions (see Fig. 8.2).

One dimension

In one dimension, the infrared divergence of the momentum integral in (8.52) suggests that there is no long-range order. An estimate of the zero-temperature correlation length can be obtained from the (crude) criterion

$$(-1)^{\mathbf{r}} \langle \hat{S}_{\mathbf{r}}^z \rangle_{T=0} \sim S - \text{const} \int_{\xi^{-1}}^{\Lambda} \frac{dq}{q} = 0, \tag{8.53}$$

expressing the fact that fluctuations above the momentum scale ξ^{-1} are responsible for the suppression of long-range order.[12] Λ denotes an upper momentum cutoff, of the order of the inverse lattice spacing, such that $1-\gamma_q \simeq q^2/2$ for $|q| \lesssim \Lambda$. Equation (8.53) yields $\xi \sim \Lambda^{-1} e^{\text{const}\times S}$, which turns out to be correct in the large-S limit for integer spins but not for half-integer spins (see Sec. 8.4.3).

[12] The momentum integrals in (8.53,8.54) stand for contributions coming from both $q \simeq 0$ and $q \simeq \pi$.

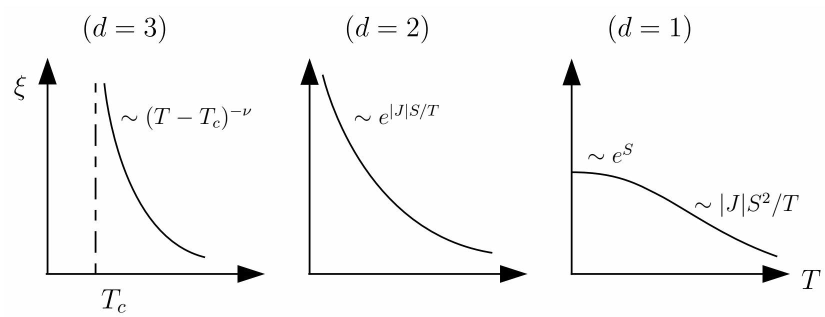

Figure 8.2: correlation length $\xi(T)$ in the Heisenberg model on a hypercubic lattice as predicted by the spinwave expansion. These predictions are qualitatively correct except for half-integer spins in one dimension.

At finite temperatures we expect the correlation length to be reduced by thermal fluctuations. The latter correspond to fluctuations with momentum smaller than T/c. If the correlation length satisfies $\xi \gg c/T$, it can be estimated from the criterion[12]

$$(-1)^{\mathbf{r}} \langle \hat{S}^z_{\mathbf{r}} \rangle_T \sim S - \frac{T}{\pi c} \int_{\xi^{-1}}^{T/c} \frac{dq}{q^2} = 0, \tag{8.54}$$

using $n_B(\omega_q) \sim T/\omega_q \sim T/c|q| \gg 1$ when $|q| \ll T/c$ ($c = 2|J|S$). When $S \gg 1$, equation (8.54) gives $\xi \sim Sc/T \sim |J|S^2/T \gg c/T$.

Two dimensions

In two dimensions, equation (8.52) gives

$$(-1)^{\mathbf{r}} \langle \hat{S}^z_{\mathbf{r}} \rangle_{T=0} \simeq S - 0.19660. \tag{8.55}$$

For $S = 1/2$, one finds $(-1)^{\mathbf{r}} \langle \hat{S}^z_{\mathbf{r}} \rangle_{T=0} \simeq 0.303$, which is in very good agreement with numerical calculations [2], somewhat surprisingly since the spinwave expansion is justified only for $S \gg 1$. To next orders, taking into account interactions between spinwaves [2],

$$(-1)^{\mathbf{r}} \langle \hat{S}^z_{\mathbf{r}} \rangle_{T=0} \simeq S - 0.19660 - \frac{0.00068}{S^2} + \mathcal{O}\left(\frac{1}{S^3}\right). \tag{8.56}$$

The vanishing of the $\mathcal{O}(1/S)$ term as well as the small coefficient of the $\mathcal{O}(1/S^2)$ may explain the success of the spinwave expansion for the Heisenberg model on a square lattice.

The infrared divergence of the momentum integral in (8.52) when $T > 0$ suggests the absence of long-range order at finite temperatures. From Eq. (8.54), with dq/q^2 replaced by $d|\mathbf{q}|/|\mathbf{q}|$, we obtain the correlation length $\xi \sim e^{\mathrm{const}\times S/T}$. This result will be confirmed in Sec. 8.4.3.

Three dimensions

In three dimensions, $(-1)^{\mathbf{r}}\langle \hat{S}^z_{\mathbf{r}}\rangle_{T=0} \simeq S - 0.07836$. The momentum integral in (8.52) remains infrared convergent at finite temperatures. At low temperatures, one finds $(-1)^{\mathbf{r}}(\langle S^z_{\mathbf{r}}\rangle - \langle S^z_{\mathbf{r}}\rangle_{T=0}) \propto T^2$. Thermodynamic quantities can easily be computed. For instance, the energy $E(T) - E(0) \propto T^4$ yields a T^3-dependent specific heat. At high temperatures, the system should be disordered by thermal fluctuations. The phase transition to the antiferromagnetic state at the Néel temperature T_N is expected to be in the universality class of the O(3) model, the correlation length $\xi \sim (T - T_N)^{-\nu}$ diverging as a power law with a critical exponent ν corresponding to three-dimensional Wilson-Fisher fixed point (chapter 10).

8.4 Spin-coherent-state functional integral

In chapter 1 we showed how to write the partition function of interacting quantum particles, be they bosons or fermions, as a functional integral. In this section we show that a similar expression of the partition function of quantum spins is possible with the help of suitably defined spin-coherent states. The spin-coherent-state functional integral allows us to recover the spinwave dynamics when the ground state exhibits spontaneously broken spin-rotation invariance. But it also enables to go beyond the spinwave analysis and show that the low-energy behavior of antiferromagnetic quantum spin systems is governed by a quantum nonlinear sigma model with a topological Berry phase term (Sec. 8.4.3), from which important results can be deduced.

8.4.1 Berry phase

Let us first discuss the concept of Berry phase, which turns out to play a crucial role in the spin-coherent-state functional integral. We consider a system with a Hamiltonian $\hat{H}(t) \equiv \hat{H}(\mathbf{R})$ depending on a a parameter $\mathbf{R} \equiv \mathbf{R}(t) = (R_1(t), R_2(t), \cdots)$ which varies with time. The evolution of the system as a function of time is given by the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}(\mathbf{R})|\psi(t)\rangle. \tag{8.57}$$

Let us assume that the instantaneous eigenstates of $\hat{H}(\mathbf{R})$ are known,

$$\hat{H}(\mathbf{R})|\psi_l(\mathbf{R})\rangle = \epsilon_l(\mathbf{R})|\psi_l(\mathbf{R})\rangle, \tag{8.58}$$

and expand $|\psi(t)\rangle$ on the basis $\{|\psi_l(\mathbf{R})\rangle\}$,

$$|\psi(t)\rangle = \sum_l c_l(t)|\psi_l(\mathbf{R})\rangle. \tag{8.59}$$

Inserting (8.59) into (8.57), we find[13]

$$i\sum_l [\dot{c}_l(t)|\psi_l(\mathbf{R})\rangle + c_l(t)\dot{\mathbf{R}}\cdot|\boldsymbol{\nabla}\psi_l(\mathbf{R})\rangle] = \sum_l c_l(t)\epsilon_l(\mathbf{R})|\psi_l(\mathbf{R})\rangle \tag{8.60}$$

[13] We use the notation $\dot{c}_l(t) = dc_l(t)/dt$ and $|\boldsymbol{\nabla}\psi_l(\mathbf{R})\rangle = \boldsymbol{\nabla}_{\mathbf{R}}|\psi_l(\mathbf{R})\rangle$.

and in turn, with a left multiplication by $\langle\psi_n(\mathbf{R})|$,

$$i\dot{c}_n(t) = c_n(t)\epsilon_n(\mathbf{R}) - \sum_l \alpha_{nl}(t)c_l(t), \tag{8.61}$$

where $\alpha_{nl}(t) = i\dot{\mathbf{R}}\cdot\langle\psi_n(\mathbf{R})|\boldsymbol{\nabla}\psi_l(\mathbf{R})\rangle$.

The adiabatic approximation

Introducing

$$\tilde{c}_n(t) = c_n(t)e^{i\int_0^t dt'\epsilon_n(\mathbf{R}')} \tag{8.62}$$

($\mathbf{R}' = \mathbf{R}(t')$), equation (8.61) becomes

$$i\dot{\tilde{c}}_n(t) = -\sum_l e^{i\int_0^t dt'\omega_{nl}(t')}\alpha_{nl}(t)\tilde{c}_l(t) \quad \text{with} \quad \omega_{nl}(t) = \epsilon_n(\mathbf{R}) - \epsilon_l(\mathbf{R}). \tag{8.63}$$

We assume that at time $t=0$, the system is in the state $|\psi_n(\mathbf{R}(0))\rangle$: $c_l(t=0) = \delta_{n,l}$. If $\mathbf{R}(t) = \mathbf{R}(0)$ is constant, then $\alpha_{nl} = 0$ and equation (8.63) gives

$$c_l(t) = \delta_{n,l}e^{-i\int_0^t dt'\epsilon_n(\mathbf{R})} \qquad (\mathbf{R}(t) = \mathbf{R}(0)). \tag{8.64}$$

When $\mathbf{R}$ varies slowly in time, we can look for a solution in powers of $\dot{\mathbf{R}}$. To lowest order,

$$i\dot{\tilde{c}}_l(t) = -e^{i\int_0^t dt'\omega_{ln}(t')}\alpha_{ln}(t) \quad \text{for} \quad l \neq n, \tag{8.65}$$

with $\tilde{c}_l(0) = 0$. If α_{ln} and ω_{ln} were time independent, equation (8.65) would give

$$\tilde{c}_l(t) = \frac{\alpha_{ln}}{\omega_{ln}}(e^{i\omega_{ln}t} - 1) \qquad (l \neq n). \tag{8.66}$$

One would then conclude that the probability amplitude $\tilde{c}_l$ remains small at any time provided that $|\alpha_{ln}/\omega_{ln}| \ll 1$. In the general case, where ω_{ln} and α_{ln} vary with time, we expect that when the condition

$$\frac{\max_t|\alpha_{ln}(t)|}{\min_t|\omega_{ln}(t)|} \ll 1 \tag{8.67}$$

is fulfilled, the system will follow adiabatically the state $|\psi_n(\mathbf{R})\rangle$, i.e. $c_n(t) \simeq 1$.[14] In that case, the time evolution of the system is entirely determined by $c_n(t)$. Note that the condition (8.67) is violated whenever the state $|\psi_n(\mathbf{R})\rangle$ becomes momentarily degenerate with another state $|\psi_l(\mathbf{R})\rangle$.

Dynamic and geometric phases

We now assume the adiabatic approximation to be valid: $|\psi(t)\rangle \simeq c_n(t)|\psi_n(\mathbf{R}\rangle$ with

$$i\dot{c}_n(t) = c_n[\epsilon_n(\mathbf{R}) - \dot{\mathbf{R}}\cdot\mathbf{A}_n(\mathbf{R})] \quad \text{and} \quad \mathbf{A}_n(\mathbf{R}) = i\langle\psi_n(\mathbf{R})|\boldsymbol{\nabla}\psi_n(\mathbf{R})\rangle, \tag{8.68}$$

[14] In a system with a large number of degrees of freedom, it is possible that $|c_l(t)|$ $(l \neq n)$ remains small even when $|c_n(t)|$ differs significantly from unity.

where the real vector $\mathbf{A}_n(\mathbf{R})$, called the Berry connection, is analog to a "vector potential".[15] Thus

$$c_n(t) = e^{-i\int_0^t dt'\epsilon_n(\mathbf{R}')+i\gamma(t)}, \tag{8.69}$$

where the first term in the exponential is the dynamic phase and the second one,

$$\gamma(t) = \int_0^t dt'\,\dot{\mathbf{R}}'\cdot\mathbf{A}_n(\mathbf{R}') = \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} d\mathbf{R}\cdot\mathbf{A}_n(\mathbf{R}), \tag{8.70}$$

the geometric (or Berry) phase. The last integral in (8.70) is taken along the line $\mathbf{R}(t')$ for $0\leq t'\leq t$. The Berry phase depends only on the "trajectory" followed by the parameter $\mathbf{R}$ and is independent of the time t to go from $\mathbf{R}(0)$ to $\mathbf{R}(t)$ (provided that the adiabatic approximation remains valid): $\gamma(t)$ is purely topological.

Since two states differing by a phase describe the same physical state, one can wonder whether the Berry phase $\gamma(t)$ carries a physical meaning or, on the contrary, can be removed by a gauge transformation. In the gauge transformation

$$|\psi_n(\mathbf{R})\rangle \to |\psi'_n(\mathbf{R})\rangle = e^{i\Lambda(\mathbf{R})}|\psi_n(\mathbf{R})\rangle, \tag{8.71}$$

the Berry connection changes as

$$\mathbf{A}_n(\mathbf{R}) \to \mathbf{A}'_n(\mathbf{R}) = i\langle\psi'_n(\mathbf{R})|\boldsymbol{\nabla}\psi'_n(\mathbf{R})\rangle = \mathbf{A}_n(\mathbf{R}) - \boldsymbol{\nabla}\Lambda(\mathbf{R}). \tag{8.72}$$

Consider now a closed path $\mathbf{R}(T)=\mathbf{R}(0)$. The Berry phase is clearly invariant since

$$\gamma(\mathcal{C}) \to \gamma'(\mathcal{C}) = \gamma(\mathcal{C}) - \oint d\mathbf{R}\cdot\boldsymbol{\nabla}\Lambda(\mathbf{R}), \tag{8.73}$$

the last integral vanishing for a closed path. Here $\mathcal{C}$ denotes the closed path $\mathbf{R}(0)\to\mathbf{R}(t)\to\mathbf{R}(T)=\mathbf{R}(0)$ and we use the notation $\gamma(\mathcal{C})$ to emphasize that the Berry phase depends only on $\mathcal{C}$ and not on the dynamics of the motion. Thus the Berry phase cannot be removed by a mere gauge transformation and has a true physical meaning as we shall see below.

When the parameter $\mathbf{R}=(R_1,R_2,R_3)$ is three-dimensional, the Berry phase

$$\gamma(\mathcal{C}) = \oint_{(\mathcal{C})} d\mathbf{R}\cdot\mathbf{A}_n(\mathbf{R}) = \int_{(\Sigma)} \mathcal{B}_n(\mathbf{R})\cdot d\boldsymbol{\Sigma}, \tag{8.74}$$

can be expressed as an integral over a surface Σ defined by the contour $\mathcal{C}$. The Berry curvature

$$\mathcal{B}_n(\mathbf{R}) = \boldsymbol{\nabla}\times\mathbf{A}_n(\mathbf{R}) = i\langle\boldsymbol{\nabla}\psi_n(\mathbf{R})|\times|\boldsymbol{\nabla}\psi_n(\mathbf{R})\rangle, \tag{8.75}$$

analog to a "magnetic field", is gauge invariant (this follows from (8.72)) like the Berry phase $\gamma(\mathcal{C})$ for a closed path, but is a local quantity.

[15] The fact that $\mathbf{A}_n(\mathbf{R})$ is real follows from $\langle\psi_n|\psi_n\rangle=1$ and $\langle\boldsymbol{\nabla}\psi_n|\psi_n\rangle+\langle\psi_n|\boldsymbol{\nabla}\psi_n\rangle=0$, so that $\langle\boldsymbol{\nabla}\psi_n|\psi_n\rangle$ is purely imaginary.

Example: spin-$\frac{1}{2}$ in a time-varying external field

We consider a spin-$\frac{1}{2}$ in an external field $\mathbf{B}(t) = B\mathbf{\Omega}(t)$ whose direction $\mathbf{\Omega}(t)$ varies slowly in time ($(\mathbf{\Omega}(t)^2 = 1$ and $B > 0$),

$$\hat{H}(t) = -B\mathbf{\Omega}(t) \cdot \hat{\mathbf{S}}, \tag{8.76}$$

with $\hat{\mathbf{S}} = \boldsymbol{\sigma}/2$ the spin operator and $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ the Pauli matrices. We assume that the adiabatic approximation is valid and that the system remains in the (instantaneous) ground state of $\hat{H}(t)$. The latter can be written as[16]

$$\begin{aligned} |\mathbf{\Omega}\rangle = \hat{U}(\mathbf{\Omega})|\uparrow\rangle &= e^{-\frac{i}{2}\varphi\sigma^z} e^{-\frac{i}{2}\theta\sigma^y} e^{-\frac{i}{2}\psi\sigma^z}|\uparrow\rangle \\ &= \cos\left(\frac{\theta}{2}\right) e^{-\frac{i}{2}(\varphi+\psi)}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right) e^{\frac{i}{2}(\varphi-\psi)}|\downarrow\rangle, \end{aligned} \tag{8.77}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle)$ denotes the eigenstates of σ^z with eigenvalue 1 and -1, respectively. θ and φ are the polar and azimuthal angles defined by the vector $\mathbf{\Omega} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, while ψ is arbitrary and depends on the gauge choice. $\hat{U}(\mathbf{\Omega})$ is a SU(2) rotation operator such that the state $|\mathbf{\Omega}\rangle$ is an eigenstate of $\hat{\mathbf{S}} \cdot \mathbf{\Omega}$ with eigenvalue 1/2.

For a closed path $\mathbf{\Omega}(T) = \mathbf{\Omega}(0)$, the Berry phase is given by (8.74) with $\mathbf{R} \equiv \mathbf{\Omega}$. An elementary calculation gives[17]

$$\mathcal{B}(\mathbf{\Omega}) = -\frac{1}{2}\mathbf{\Omega} \quad \text{and} \quad \gamma(\mathcal{C}) = -\frac{1}{2}\int_{(\Sigma)} \mathbf{\Omega} \cdot d\mathbf{\Sigma} = -\frac{1}{2}\omega(\mathcal{C}), \tag{8.78}$$

where $\omega(\mathcal{C})$ is the oriented solid angle (seen from the origin) defined by the closed path $\mathcal{C}$ on the unit sphere. This expression of the "magnetic field" $\mathcal{B}$ shows that the latter is not created by a current distribution but by a magnetic monopole located at the origin. This monopole has a charge q_m defined by $\boldsymbol{\nabla} \cdot \mathcal{B} = 4\pi q_m \delta(\mathbf{B})$, i.e. $q_m = -\frac{1}{2}$.[18,19] Note that there is an ambiguity in the expression (8.78) of the Berry phase $\gamma(\mathcal{C})$ since there are two possible choices for the surface Σ corresponding to the interior of the closed path $\mathcal{C}$ on the unit sphere. However, since the two solid angles differ by 4π, the resulting Berry phases differ by 2π and are therefore equivalent.

There is actually a subtle point here. The state $|\mathbf{\Omega}\rangle$ is not defined everywhere on the sphere. For instance, in the gauge $\psi = -\varphi$, $|\mathbf{\Omega}\rangle$ is not defined at the south pole since $\sin(\frac{\theta}{2})e^{i\varphi}$ does not approach a unique value when $\theta \to \pi$. Thus the Berry connection $\mathbf{A}(\mathbf{\Omega})$ and the Berry curvature $\mathcal{B}(\mathbf{\Omega})$ are defined everywhere on the unit sphere except at the south pole. An explicit calculation gives[17]

$$\mathbf{A}(\mathbf{\Omega}) = i\langle\mathbf{\Omega}|\boldsymbol{\nabla}\mathbf{\Omega}\rangle = -\frac{1-\cos\theta}{2\sin\theta}\mathbf{u}_\varphi, \tag{8.79}$$

where $\mathbf{u}_\varphi = (-\sin\varphi, \cos\varphi, 0)$. From (8.79), one easily sees that the line integral of the vector potential along a tiny loop around the south pole,

$$\lim_{\theta\to\pi} \int_0^{2\pi} d\varphi\, \mathbf{A}(\mathbf{\Omega}) \cdot \mathbf{u}_\varphi = \lim_{\theta\to\pi} A_\varphi 2\pi \sin\theta = -2\pi, \tag{8.80}$$

[16] $|\mathbf{\Omega}\rangle$ is an example of spin-coherent states; see Sec. 8.4.2.

[17] We use spherical coordinates to compute $\mathcal{B}(\mathbf{\Omega})$ and $\mathbf{A}(\mathbf{\Omega})$.

[18] If, more generally, we had also allowed the amplitude of the external field $\mathbf{B}(t) = B(t)\mathbf{\Omega}(t)$ to vary in time, we would have found a Berry curvature $\mathcal{B}(\mathbf{B}) = -\mathbf{B}/(2\mathbf{B}^3)$.

[19] Here we use CGS units. In SI units $\boldsymbol{\nabla} \cdot \mathcal{B} = q_m\delta(\mathbf{B})$ or $\boldsymbol{\nabla} \cdot \mathcal{B} = \mu_0 q_m \delta(\mathbf{B})$.

gives the full monopole flux $4\pi q_m$. Thus the "vector potential" $\mathbf{A}$ describes not a monopole at the origin but a tiny flux line (the Dirac string) along the negative-z axis, which brings the entire flux to the origin from where it emanates radially.[20] Using (8.79) one recovers the result (8.78),

$$\gamma(\mathcal{C}) = \oint_{(\mathcal{C})} d\mathbf{\Omega} \cdot \mathbf{A}(\mathbf{\Omega}) = \int_0^T dt \frac{\dot{\varphi}}{2}(\cos\theta - 1) \equiv -\frac{1}{2}\omega(\mathcal{C}), \tag{8.81}$$

where now $\omega(\mathcal{C})$ is defined such as excluding the south pole.

The location of the Dirac string depends on the definition of $|\mathbf{\Omega}\rangle$. In the gauge $\psi = \varphi$, the Berry connection[21,22]

$$\mathbf{A}'(\mathbf{\Omega}) = \frac{1+\cos\theta}{2\sin\theta}\mathbf{u}_\varphi \tag{8.82}$$

is not defined at the north pole. The Berry phase reads

$$\gamma'(\mathcal{C}) = \oint_{(\mathcal{C})} d\mathbf{\Omega} \cdot \mathbf{A}'(\mathbf{\Omega}) = \int_0^T dt \frac{\dot{\varphi}}{2}(\cos\theta + 1) = -\frac{1}{2}\omega'(\mathcal{C}) = \gamma(\mathcal{C}) + 2\pi, \tag{8.83}$$

where $\omega'(\mathcal{C}) = \omega(\mathcal{C}) - 4\pi$ is defined by the surface Σ' excluding the north pole.

The preceding analysis can be generalized to an arbitrary spin S (see Sec. 8.4.2). Again one can use different gauges to calculate the Berry phase. For instance, equations (8.81) and (8.83) become $\gamma(\mathcal{C}) = -S\omega(\mathcal{C})$ and $\gamma'(\mathcal{C}) = \gamma(\mathcal{C}) + 4S\pi$. Demanding that $\gamma(\mathcal{C})$ be defined modulo 2π leads to the familiar requirement that $2S$ be an integer.

8.4.2 Spin-coherent-state functional integral

Spin-coherent states

We consider a spin S and denote by $|S^z = -S, \cdots, S\rangle$ the $2S+1$ eigenstates of $\hat{S}^z$. Spin-coherent states are defined by

$$|\mathbf{\Omega}\rangle = e^{-i\varphi\hat{S}^z} e^{-i\theta\hat{S}^y} e^{-i\psi\hat{S}^z} |S\rangle, \tag{8.84}$$

where $\mathbf{\Omega}(\theta,\varphi)$ is the unit vector introduced in the preceding section. $|\mathbf{\Omega}\rangle$ is obtained from $|S\rangle$ by a rotation of angle θ about the y axis followed by a rotation of angle φ about the z axis. The angle ψ is arbitrary and corresponds to a gauge freedom (Sec. 8.4.1). Since $\hat{\mathbf{S}}$ transforms like a vector in a spin rotation (Sec. 2.2.3),

$$\langle\mathbf{\Omega}|\hat{\mathbf{S}}|\mathbf{\Omega}\rangle = S\mathbf{\Omega}. \tag{8.85}$$

Note however that $|\mathbf{\Omega}\rangle$ is not an eigenstate of $\hat{\mathbf{S}}$. In Appendix 8.B, we show that the scalar product between two coherent states is given by

$$\langle\mathbf{\Omega}'|\mathbf{\Omega}\rangle = \left[\frac{1}{2} + \frac{1}{2}\mathbf{\Omega}'\cdot\mathbf{\Omega}\right]^S, \tag{8.86}$$

[20] Recall that any vector potential describing a monopole has a string.

[21] $\mathbf{A}$ and $\mathbf{A}'$ [Eqs. (8.79,8.82)] are related by the gauge transformation $|\mathbf{\Omega}'\rangle = e^{\frac{i}{2}\varphi}|\mathbf{\Omega}\rangle$.

[22] Another common gauge choice is $\psi = 0$, which gives $\mathbf{A}(\mathbf{\Omega}) = \frac{1}{2\tan\theta}\mathbf{u}_\varphi$.

and obtain the following resolution of the identity,

$$(2S+1)\int \frac{d\Omega}{4\pi} |\mathbf{\Omega}\rangle\langle\mathbf{\Omega}| = 1, \tag{8.87}$$

where the "1" in the rhs of (8.87) denotes the identity operator.

Spin-coherent-state functional integral

Let us consider a single spin S in an external field with Hamiltonian

$$\hat{H} = -\mathbf{h}\cdot\hat{\mathbf{S}}. \tag{8.88}$$

We can write the partition function

$$Z = \mathrm{Tr}\, e^{-\beta\hat{H}} = (2S+1)\int \frac{d\Omega}{4\pi}\langle\mathbf{\Omega}|e^{-\beta\hat{H}}|\mathbf{\Omega}\rangle \tag{8.89}$$

as a functional integral over spin-coherent states following the general method described in chapter 1. First we split the imaginary "time" β into M infinitesimal steps $\epsilon = \beta/M$ and introduce $M-1$ times the resolution of the identity (8.87),

$$Z = \left(\frac{2S+1}{4\pi}\right)^M \int \prod_{k=1}^M d\Omega_k \prod_{k=1}^M \langle\mathbf{\Omega}_k|e^{-\epsilon\hat{H}}|\mathbf{\Omega}_{k-1}\rangle, \tag{8.90}$$

with $\Omega_0 = \Omega_M$. We then use

$$\begin{aligned}\langle\mathbf{\Omega}_k|e^{-\epsilon\hat{H}}|\mathbf{\Omega}_{k-1}\rangle &= \langle\mathbf{\Omega}_k|1-\epsilon\hat{H}|\mathbf{\Omega}_{k-1}\rangle + \mathcal{O}(\epsilon^2) \\ &= \langle\mathbf{\Omega}_k|\mathbf{\Omega}_{k-1}\rangle - \epsilon\langle\mathbf{\Omega}_k|\hat{H}|\mathbf{\Omega}_{k-1}\rangle + \mathcal{O}(\epsilon^2). \end{aligned} \tag{8.91}$$

Setting $\tau = k\epsilon$ and considering $\tau \in [0,\beta]$ as a continuous variable in the limit $\epsilon = \beta/M \to 0$, we finally obtain[23,24]

$$\begin{aligned}\langle\mathbf{\Omega}_k|e^{-\epsilon\hat{H}}|\mathbf{\Omega}_{k-1}\rangle &= \langle\mathbf{\Omega}|\mathbf{\Omega}\rangle - \epsilon\langle\mathbf{\Omega}|\partial_\tau\mathbf{\Omega}\rangle - \epsilon\langle\mathbf{\Omega}|\hat{H}|\mathbf{\Omega}\rangle + \mathcal{O}(\epsilon^2) \\ &= e^{-\epsilon\langle\mathbf{\Omega}|\partial_\tau\mathbf{\Omega}\rangle - \epsilon H[\mathbf{\Omega}]} + \mathcal{O}(\epsilon^2), \end{aligned} \tag{8.92}$$

where $H[\mathbf{\Omega}] = \langle\mathbf{\Omega}|\hat{H}|\mathbf{\Omega}\rangle$ and we have used $\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle = 1$. Inserting (8.92) into (8.90) gives

$$\begin{aligned} Z &= \left(\frac{2S+1}{4\pi}\right)^M \int \prod_{k=1}^M d\Omega_k \exp\left\{-\epsilon\sum_{k=1}^M \big(\langle\mathbf{\Omega}|\partial_\tau\mathbf{\Omega}\rangle + H[\mathbf{\Omega}]\big)\right\} \\ &\equiv \int_{\mathbf{\Omega}(\beta)=\mathbf{\Omega}(0)} \mathcal{D}[\mathbf{\Omega}] \exp\left\{-\mathcal{S}_B[\mathbf{\Omega}] - \int_0^\beta d\tau\, H[\mathbf{\Omega}]\right\}, \end{aligned} \tag{8.93}$$

[23] We use the notation $|\partial_\tau\mathbf{\Omega}\rangle = \partial_\tau|\mathbf{\Omega}\rangle$.

[24] As pointed out in chapter 1, the continuous-time limit in a functional integral may be ill-defined and in case of doubt one should always return to the original discrete-time formulation.

where the action

$$\mathcal{S}_B[\mathbf{\Omega}] = \int_0^\beta d\tau\, \langle \mathbf{\Omega}|\partial_\tau \mathbf{\Omega}\rangle = -i \int_0^\beta d\tau\, \dot{\mathbf{\Omega}} \cdot \mathbf{A}(\mathbf{\Omega}) \tag{8.94}$$

is the Berry phase term introduced in section 8.4.1 in the particular case $S = \frac{1}{2}$. To compute the "vector potential" $\mathbf{A}(\mathbf{\Omega}) = i\langle \mathbf{\Omega}|\boldsymbol{\nabla}\mathbf{\Omega}\rangle$ for an arbitrary value of S, we use

$$|\boldsymbol{\nabla}\mathbf{\Omega}\rangle = \mathbf{u}_\theta \frac{\partial}{\partial\theta}|\mathbf{\Omega}\rangle + \mathbf{u}_\varphi \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}|\mathbf{\Omega}\rangle \tag{8.95}$$

and

$$\langle \mathbf{\Omega}|\partial_\theta \mathbf{\Omega}\rangle = 0, \quad \langle \mathbf{\Omega}|\partial_\varphi \mathbf{\Omega}\rangle = iS(1-\cos\theta), \tag{8.96}$$

where $\mathbf{u}_\varphi = (-\sin\varphi, \cos\varphi, 0)$ and $\mathbf{u}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$ (see Appendix 8.B). We thus obtain

$$\begin{aligned} \mathbf{A}(\mathbf{\Omega}) &= -S\frac{1-\cos\theta}{\sin\theta}\mathbf{u}_\varphi, \\ \mathcal{S}_B[\mathbf{\Omega}] &= iS\int_0^\beta d\tau\, \dot{\varphi}(1-\cos\theta), \end{aligned} \tag{8.97}$$

which agrees with (8.79) for $S = 1/2$ and the gauge choice $\psi = -\varphi$. Using the results of section 8.4.1, we conclude that the Berry phase term

$$\mathcal{S}_B[\mathbf{\Omega}] = -i\int_0^\beta d\tau\, \dot{\mathbf{\Omega}}\cdot\mathbf{A}(\mathbf{\Omega}) = iS\omega(\mathcal{C}) \tag{8.98}$$

is simply determined by the solid angle $\omega(\mathcal{C})$ corresponding to the closed path $\mathcal{C}$: $\mathbf{\Omega}(0) \to \mathbf{\Omega}(\beta)$ on the unit sphere.

We are now in a position to determine the classical spin dynamics of a single spin in an external field $\mathbf{h}$:

$$\frac{\delta}{\delta\mathbf{\Omega}(\tau)}\left\{\mathcal{S}_B[\mathbf{\Omega}] - S\int_0^\beta d\tau\, \mathbf{h}\cdot\mathbf{\Omega}(\tau)\right\} = 0. \tag{8.99}$$

To compute the first functional derivative, we consider the variation of the Berry phase term (8.94) due to a change $\delta\mathbf{\Omega}(\tau)$,

$$\delta\mathcal{S}_B = -i\int_0^\beta d\tau \left(\delta\dot{\Omega}_\alpha A_\alpha + \dot{\Omega}_\alpha \frac{\partial A_\alpha}{\partial\Omega_\gamma}\delta\Omega_\gamma\right) \tag{8.100}$$

(with an implicit sum over repeated indices). Adding and subtracting $(\partial A_\alpha/\partial\Omega_\gamma)\dot{\Omega}_\gamma\delta\Omega_\alpha$, we obtain

$$\delta\mathcal{S}_B = -i\int_0^\beta d\tau \left(\frac{\partial A_\alpha}{\partial\Omega_\gamma}(\dot{\Omega}_\alpha\delta\Omega_\gamma - \dot{\Omega}_\gamma\delta\Omega_\alpha) + \delta\dot{\Omega}_\alpha A_\alpha + \frac{\partial A_\alpha}{\partial\Omega_\gamma}\dot{\Omega}_\gamma\delta\Omega_\alpha\right). \tag{8.101}$$

The last two terms give $\partial_\tau(A_\alpha\delta\Omega_\alpha)$ and their contribution vanishes because of the periodic boundary conditions in imaginary time. Thus

$$\delta\mathcal{S}_B = -i\int_0^\beta d\tau\, \epsilon_{\alpha\gamma\gamma'}\frac{\partial A_\alpha}{\partial\Omega_\gamma}(\dot{\mathbf{\Omega}}\times\delta\mathbf{\Omega})_{\gamma'} = i\int_0^\beta d\tau (\boldsymbol{\nabla}\times\mathbf{A})\cdot(\dot{\mathbf{\Omega}}\times\delta\mathbf{\Omega}). \tag{8.102}$$

Using the fact that the Berry connection $\mathcal{B} = \boldsymbol{\nabla} \times \mathbf{A} = -S\boldsymbol{\Omega}$ (Sec. 8.4.1), we finally obtain

$$\delta \mathcal{S}_B = -iS \int_0^\beta d\tau \, \delta\boldsymbol{\Omega} \cdot (\boldsymbol{\Omega} \times \dot{\boldsymbol{\Omega}}). \tag{8.103}$$

The classical equation of motion (8.99) then reads

$$i\dot{\boldsymbol{\Omega}} = \boldsymbol{\Omega} \times \mathbf{h}. \tag{8.104}$$

Performing a Wick rotation to real time, $t = -i\tau$, we recover the classical equation of motion of a spin $\mathbf{S} = S\boldsymbol{\Omega}$ in an external field: $d\mathbf{S}/dt = \mathbf{S} \times \mathbf{h}$ [Eq. (8.19)].

The generalization of the spin-coherent-state functional integral to a many-spin Hamiltonian defined on a lattice is straightforward. Spin-coherent states $|\boldsymbol{\Omega}\rangle$ are now defined by a unit vector field $\boldsymbol{\Omega}_\mathbf{r}$ defined at each lattice site $\mathbf{r}$, and the resolution of the identity becomes

$$\left(\frac{2S+1}{4\pi}\right)^N \int \prod_\mathbf{r} d\boldsymbol{\Omega}_\mathbf{r} |\boldsymbol{\Omega}\rangle\langle\boldsymbol{\Omega}| = 1. \tag{8.105}$$

The partition function can be written as a functional integral

$$Z = \int_{\boldsymbol{\Omega}(\beta)=\boldsymbol{\Omega}(0)} \mathcal{D}[\boldsymbol{\Omega}] \, e^{-\mathcal{S}_B[\boldsymbol{\Omega}] - \int_0^\beta d\tau H[\boldsymbol{\Omega}]}, \tag{8.106}$$

where the Berry phase term

$$\mathcal{S}_B[\boldsymbol{\Omega}] = \int_0^\beta d\tau \langle\boldsymbol{\Omega}|\partial_\tau\boldsymbol{\Omega}\rangle = \int_0^\beta d\tau \sum_\mathbf{r} \langle\boldsymbol{\Omega}_\mathbf{r}|\partial_\tau\boldsymbol{\Omega}_\mathbf{r}\rangle \tag{8.107}$$

is simply the sum of the single-spin contributions. $H[\boldsymbol{\Omega}] = \langle\boldsymbol{\Omega}|\hat{H}|\boldsymbol{\Omega}\rangle$, e.g.

$$H[\boldsymbol{\Omega}] = -\frac{S^2}{2} \sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} \boldsymbol{\Omega}_\mathbf{r} \cdot \boldsymbol{\Omega}_{\mathbf{r}'} \tag{8.108}$$

for the Heisenberg model (8.1), is obtained using standard properties of the spin-coherent states. The equation of motion $\delta S/\delta\boldsymbol{\Omega}_\mathbf{r}(\tau) = 0$ reproduces the classical dynamics, i.e.

$$i\dot{\boldsymbol{\Omega}}_\mathbf{r} = S \sum_{\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} \boldsymbol{\Omega}_\mathbf{r} \times \boldsymbol{\Omega}_{\mathbf{r}'} \tag{8.109}$$

in real time. In section 8.2.1 we have seen that linearizing this equation about the classical solution $\boldsymbol{\Omega}^{\rm cl}_\mathbf{r} = \mathbf{u}_z$ (ferromagnet) or $\boldsymbol{\Omega}^{\rm cl}_\mathbf{r} = (-1)^\mathbf{r}\mathbf{u}_z$ (antiferromagnet) yields the spinwave excitations. Spinwaves are discussed in more detail below.

Spin-waves

In the limit $S \to \infty$, the action is of order S^2 and a saddle-point approximation becomes exact.[25] The variable $\boldsymbol{\Omega}_\mathbf{r}(\tau)$ does not fluctuate and coincides with the classical

[25] Formally we can justify the saddle-point approximation and the loop expansion by rescaling the time variable: $\tau \to \bar\tau = S\tau$ and $\beta \to \bar\beta = S\beta$. The action $\mathcal{S}[\boldsymbol{\Omega}]$ then becomes proportional to S and a loop expansion, controlled by the small parameter $1/S$, becomes possible (Sec. 1.7).

solution $\mathbf{\Omega}_{\mathbf{r}}^{\rm cl}$ obtained by minimizing the action,

$$\frac{\delta\mathcal{S}[\mathbf{\Omega}]}{\delta\mathbf{\Omega}_{\mathbf{r}}(\tau)}\bigg|_{\mathbf{\Omega}_{\mathbf{r}}(\tau)=\mathbf{\Omega}_{\mathbf{r}}^{\rm cl}} = 0. \tag{8.110}$$

In the large-S limit, we expect the functional integral to be dominated by small fluctuations $\delta\mathbf{\Omega}_{\mathbf{r}} = \mathbf{\Omega}_{\mathbf{r}} - \mathbf{\Omega}_{\mathbf{r}}^{\rm cl}$ about the classical state. We use the parameterization

$$P_{\mathbf{r}} = S\delta\mathbf{\Omega}_{\mathbf{r}}\cdot\mathbf{u}_\theta, \qquad Q_{\mathbf{r}} = \delta\mathbf{\Omega}_{\mathbf{r}}\cdot\mathbf{u}_\varphi. \tag{8.111}$$

For small fluctuations, the spin-coherent-state functional integral becomes an integral over $P_{\mathbf{r}}$ and $Q_{\mathbf{r}}$, i.e. $\mathcal{D}[\mathbf{\Omega}]\to\mathcal{D}[P,Q]$, and the domain of integration over $P_{\mathbf{r}}$ and $Q_{\mathbf{r}}$ can be extended to $]-\infty,\infty[$. We shall see that the spinwave dynamics is encoded in the action $\mathcal{S}[P,Q]$ to quadratic order.

Let us first consider the ferromagnetic case with $\mathbf{\Omega}_{\mathbf{r}}^{\rm cl} = \mathbf{u}_x$. One then has

$$P_{\mathbf{r}} = -S\cos\theta_{\mathbf{r}} \quad \text{and} \quad Q_{\mathbf{r}} = \sin\theta_{\mathbf{r}}\sin\varphi_{\mathbf{r}} \simeq \varphi_{\mathbf{r}}. \tag{8.112}$$

To determine the Berry phase term to quadratic order in P,Q, we use (8.94), which gives[26,27]

$$\mathcal{S}_B = -i\frac{S}{2}\int_0^\beta d\tau \sum_{\mathbf{r}} \mathbf{\Omega}_{\mathbf{r}}^{\rm cl}\cdot(\delta\dot{\mathbf{\Omega}}_{\mathbf{r}}\times\delta\mathbf{\Omega}_{\mathbf{r}}) = \frac{i}{2}\int_0^\beta d\tau\sum_{\mathbf{r}}(P_{\mathbf{r}}\dot{Q}_{\mathbf{r}} - \dot{P}_{\mathbf{r}}Q_{\mathbf{r}}) \tag{8.113}$$

to quadratic order. As for the Hamiltonian, we obtain

$$\begin{aligned} H[\mathbf{\Omega}] &= -JS^2\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}\left[\cos\theta_{\mathbf{r}}\cos\theta_{\mathbf{r}'} + \sin\theta_{\mathbf{r}}\sin\theta_{\mathbf{r}'}\cos(\varphi_{\mathbf{r}}-\varphi_{\mathbf{r}'})\right] \\ &= E_{\rm cl} + \frac{JS^2}{2}\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}\left[\frac{(P_{\mathbf{r}}-P_{\mathbf{r}'})^2}{S^2} + (Q_{\mathbf{r}}-Q_{\mathbf{r}'})^2\right] \end{aligned} \tag{8.114}$$

to quadratic order in P and Q. In Fourier space, we can therefore write the action as (dropping the ferromagnetic ground state energy $E_{\rm cl}$)

$$\mathcal{S}[P,Q] = \frac{1}{2}\sum_q (P_{-q},Q_{-q})\begin{pmatrix} 2Jd(1-\gamma_{\mathbf{q}}) & \omega_\nu \\ -\omega_\nu & 2JdS^2(1-\gamma_{\mathbf{q}})\end{pmatrix}\begin{pmatrix} P_q \\ Q_q\end{pmatrix}, \tag{8.115}$$

with $q=(\mathbf{q},i\omega_\nu)$ and $\gamma_{\mathbf{q}}$ defined in (8.42). The dispersion of the spinwave modes is obtained from the vanishing of the determinant of the 2×2 matrix in (8.115) after analytical continuation $i\omega_\nu\to\omega+i0^+$, i.e. $\omega_{\mathbf{q}} = 2dJS(1-\gamma_{\mathbf{q}})$, in agreement with the results of section 8.3.1.

In the antiferromagnetic case ($J<0$) we choose $\mathbf{\Omega}_{\mathbf{r}}^{\rm cl} = (-1)^{\mathbf{r}}\mathbf{u}_x$ so that

$$P_{\mathbf{r}} = -S\cos\theta_{\mathbf{r}} \quad \text{and} \quad Q_{\mathbf{r}} = \begin{cases} \sin\theta_{\mathbf{r}}\sin\varphi_{\mathbf{r}} \simeq \varphi_{\mathbf{r}} & \text{if} \quad \mathbf{r}\in A, \\ -\sin\theta_{\mathbf{r}}\sin\varphi_{\mathbf{r}} \simeq \varphi_{\mathbf{r}}+\pi & \text{if} \quad \mathbf{r}\in B. \end{cases} \tag{8.116}$$

[26] Eq. (8.113) follows from (for a single site and with an implicit sum over repeated indices): $\mathcal{S}_B \simeq -i\int_0^\beta d\tau\delta\dot{\mathbf{\Omega}}_\alpha \frac{\partial A_\alpha}{\partial\mathbf{\Omega}_\gamma}\big|_{\rm cl}\delta\mathbf{\Omega}_\gamma = -\frac{i}{2}\int_0^\beta d\tau\frac{\partial A_\alpha}{\partial\mathbf{\Omega}_\gamma}\big|_{\rm cl}(\delta\dot{\mathbf{\Omega}}_\alpha\delta\mathbf{\Omega}_\gamma - \delta\mathbf{\Omega}_\alpha\delta\dot{\mathbf{\Omega}}_\gamma) = \frac{i}{2}\int_0^\beta d\tau(\boldsymbol{\nabla}\times\mathbf{A})\cdot(\delta\dot{\mathbf{\Omega}}\times\delta\mathbf{\Omega})$ with $\boldsymbol{\nabla}\times\mathbf{A} = -S\mathbf{\Omega}^{\rm cl}$.

[27] Alternatively, Eq. (8.113) can be obtained from an explicit expression of $\mathbf{A}(\mathbf{\Omega})$ and $\mathcal{S}_B[\mathbf{\Omega}]$ in a given gauge, e.g. Eq. (8.97).

The Berry phase term is unchanged [Eq. (8.113)] and the Hamiltonian is given by

$$H[\mathbf{\Omega}] = E_{\rm cl} + \frac{|J|S^2}{2} \sum_{\langle \mathbf{r},\mathbf{r}'\rangle} \left[\frac{(P_\mathbf{r} + P_{\mathbf{r}'})^2}{S^2} + (Q_\mathbf{r} - Q_{\mathbf{r}'})^2 \right] \tag{8.117}$$

to quadratic order in P and Q. This yields the action (doping the energy $E_{\rm cl}$ of the classical antiferromagnetic state)

$$\mathcal{S}[P,Q] = \frac{1}{2} \sum_q (P_{-q}, Q_{-q}) \begin{pmatrix} 2|J|d(1+\gamma_\mathbf{q}) & \omega_\nu \\ -\omega_\nu & 2|J|dS^2(1-\gamma_\mathbf{q}) \end{pmatrix} \begin{pmatrix} P_q \\ Q_q \end{pmatrix}, \tag{8.118}$$

and in turn the spinwave dispersion $\omega_\mathbf{q} = 2d|J|S(1-\gamma_\mathbf{q}^2)^{1/2}$, in agreement with the results of section 8.3.2.

8.4.3 Quantum nonlinear sigma model

The spinwave expansion is valid provided that there is long-range order. In this section we consider a more general situation where the system exhibits short-range antiferromagnetic order. The antiferromagnetic correlation length ξ is then much larger than the lattice spacing but not necessary infinite as it would be in the presence of long-range order. We expect spinwave fluctuations about the local antiferromagnetic order to be the dominant fluctuations at length scales smaller than ξ. We shall see that the low-energy behavior is described by a quantum nonlinear sigma model with a Berry phase term.[28]

To capture short-range antiferromagnetic order it is tempting to write $\mathbf{\Omega}_\mathbf{r} \simeq (-1)^\mathbf{r} \mathbf{n}_\mathbf{r}$ where the Néel field $\mathbf{n}_\mathbf{r}$ is expected to be slowly varying. But the spinwave expansion about the classical Néel state shows that antiferromagnetic fluctuations with momentum near $\mathbf{Q} = (\pi, \cdots, \pi)$ couple to long-wavelength ferromagnetic fluctuations.[29] Thus short-range antiferromagnetic order is captured by the following parameterization[30]

$$\mathbf{\Omega}_\mathbf{r} = (-1)^\mathbf{r} \mathbf{n}_\mathbf{r} \left(1 - \frac{\mathbf{L}_\mathbf{r}^2}{S^2} \right)^{1/2} + \frac{\mathbf{L}_\mathbf{r}}{S}, \tag{8.119}$$

where

$$\mathbf{n}_\mathbf{r}^2 = 1 \quad \text{and} \quad \mathbf{n}_\mathbf{r} \perp \mathbf{L}_\mathbf{r} = 0, \tag{8.120}$$

so that $\mathbf{\Omega}_\mathbf{r}^2 = 1$. Both the Néel field $\mathbf{n}_\mathbf{r}$ and the canting field $\mathbf{L}_\mathbf{r}$, which describes ferromagnetic fluctuations of the spins, are assumed to be slowly varying (an assumption that should hold at least in the large-S limit). The prefactor $1/S$ has been associated with $\mathbf{L}_\mathbf{r}$ so that the spatial integral of $\mathbf{L}_\mathbf{r}$ over any region is precisely the total magnetization in that region. Moreover $\mathbf{L}_\mathbf{r}$ is assumed to be small, $|\mathbf{L}_\mathbf{r}|/S \ll 1$, so that spins on nearby sites are predominantly antiparallel.

[28]For a general discussion of the quantum nonlinear sigma model, see section 12.3.

[29]In Eq. (8.118), fluctuations $P_{\mathbf{q}\simeq 0}$ and $Q_{\mathbf{q}\simeq 0}$ correspond to ferromagnetic fluctuations along u_z and antiferromagnetic fluctuations along u_y, respectively (the reverse is true for $\mathbf{q} \simeq \mathbf{Q}$).

[30]We shall see that $\mathbf{L}_\mathbf{r}$ and $\mathbf{n}_\mathbf{r} \times \dot{\mathbf{n}}_\mathbf{r}$ are conjugated fields [Eq. (8.128)], which shows *a posteriori* the necessity to introduce $\mathbf{L}_\mathbf{r}$.

We can therefore look for an effective action $\mathcal{S}[\mathbf{n},\mathbf{L}]$ in the continuum limit and to leading order in a derivative expansion. This is done by writing the partition function as a functional integral over the fields $\mathbf{n_r}$ and $\mathbf{L_r}$ using

$$\mathcal{D}[\boldsymbol{\Omega}] = \mathcal{D}[\mathbf{n},\mathbf{L}]\mathcal{J}[\mathbf{n},\mathbf{L}]\prod_{\mathbf{r}}\delta(\mathbf{n}_{\mathbf{r}}^2-1)\delta(\mathbf{n_r}\cdot\mathbf{L_r}), \tag{8.121}$$

where $\mathcal{J}[\mathbf{n},\mathbf{L}]$, the Jacobian of the transformation (8.119), is a constant to leading order in $1/S$.[31] In the transformation (8.119) it seems that we have replaced 2 degrees of freedom per site (e.g. the polar and azimuthal angles, $\theta_{\mathbf{r}}$ and $\varphi_{\mathbf{r}}$, of the spin variable $\boldsymbol{\Omega}_{\mathbf{r}}$) by 4 (6 degrees of freedom for $\mathbf{n_r}$ and $\mathbf{L_r}$ minus the 2 constraints (8.120)). This increase in the number of degrees of freedom is due to the decomposition (8.119) being not unique; nevertheless it has no effect on the low-energy effective action of the Néel field we wish to derive.[32] To rewrite the Heisenberg Hamiltonian in terms of $\mathbf{n_r}$ and $\mathbf{L_r}$ we use

$$\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}\boldsymbol{\Omega}_{\mathbf{r}}\cdot\boldsymbol{\Omega}_{\mathbf{r}'} = \frac{1}{2}\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}(\boldsymbol{\Omega}_{\mathbf{r}}+\boldsymbol{\Omega}_{\mathbf{r}'})^2 + \text{const} \tag{8.122}$$

and

$$\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}(\boldsymbol{\Omega}_{\mathbf{r}}+\boldsymbol{\Omega}_{\mathbf{r}'})^2 \simeq \sum_{\langle\mathbf{r},\mathbf{r}'\rangle}\left[(\mathbf{n_r}-\mathbf{n_{r'}})^2 + \frac{1}{S^2}(\mathbf{L_r}+\mathbf{L_{r'}})^2\right] \tag{8.123}$$

if we ignore terms of order $|\mathbf{L_r}|^3$, $\mathbf{L}_{\mathbf{r}}^2(\mathbf{n_r}-\mathbf{n_{r'}})$ and $(\mathbf{L_r}+\mathbf{L_{r'}})\cdot(\mathbf{n_r}-\mathbf{n_{r'}})$.[33]. The last term in (8.123) gives a contribution to the Hamiltonian

$$-\frac{J}{2}\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}(\mathbf{L_r}+\mathbf{L_{r'}})^2 = -\frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'}\mathbf{L_r}\cdot\mathbf{L_{r'}}(2dJ\delta_{\mathbf{r},\mathbf{r}'}+J_{\mathbf{r},\mathbf{r}'}) \equiv \frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'}\chi^{-1}_{\perp\mathbf{r},\mathbf{r}'}\mathbf{L_r}\cdot\mathbf{L_{r'}}, \tag{8.124}$$

where

$$\chi_\perp(\mathbf{q}) = \frac{1}{J(\mathbf{Q})-J(\mathbf{q})} \tag{8.125}$$

is the static ($\omega=0$) transverse susceptibility in the Néel state obtained in the RPA.[34] To compute the Berry phase term we use equation (8.103) with $\boldsymbol{\Omega}_{\mathbf{r}} = (-1)^{\mathbf{r}}\mathbf{n_r}$ and $\delta\boldsymbol{\Omega}_{\mathbf{r}} = \mathbf{L_r}/S + \mathcal{O}(\mathbf{L}_{\mathbf{r}}^2)$,

$$\mathcal{S}_B = \mathcal{S}'_B[\mathbf{n}] - i\int_0^\beta d\tau\sum_{\mathbf{r}}\mathbf{L_r}\cdot(\mathbf{n_r}\times\partial_\tau\mathbf{n_r}), \tag{8.126}$$

where

$$\mathcal{S}'_B[\mathbf{n}] = \int_0^\beta d\tau\sum_{\mathbf{r}}(-1)^{\mathbf{r}}\langle\mathbf{n_r}|\partial_\tau\mathbf{n_r}\rangle \tag{8.127}$$

[31] In the large-S limit, we can parameterize the small fluctuations of $\mathbf{n_r}$ and $\mathbf{L_r}$ by $P_{\mathbf{r}}$ and $Q_{\mathbf{r}}$ [Eqs. (8.111)] and let these variables vary between $-\infty$ and ∞, which shows that the Jacobian is a mere S-dependent constant.

[32] We could impose the correct number of degrees of freedom by reducing the Brillouin zone by a factor of 2.

[33] In the continuum limit these terms are $\mathcal{O}(|\mathbf{L}|^3, \mathbf{L}^2\boldsymbol{\nabla}\mathbf{n}, \mathbf{L}\cdot\boldsymbol{\nabla}^2\mathbf{n})$.

[34] The interpretation of $\chi_\perp(\mathbf{q})$ as the transverse susceptibility in the Néel state follows from $\chi^{-1} = \chi_0^{-1} - J$ in the RPA (in a matrix sense) with χ_0 defined by (8.36).

is the Berry phase term to zeroth order in $\mathbf{L}$.

We thus obtain the action

$$\begin{aligned} \mathcal{S}[\mathbf{n},\mathbf{L}] = \mathcal{S}'_B[\mathbf{n}] - i\int_0^\beta d\tau \sum_{\mathbf{r}} \mathbf{L}_\mathbf{r}\cdot(\mathbf{n}_\mathbf{r}\times\partial_\tau\mathbf{n}_\mathbf{r}) \\ + \frac{1}{2}\int_0^\beta d\tau \sum_{\langle\mathbf{r},\mathbf{r}'\rangle} \Big[|J|S^2(\mathbf{n}_\mathbf{r}-\mathbf{n}_{\mathbf{r}'})^2 + \chi^{-1}_{\perp\mathbf{r},\mathbf{r}'}\mathbf{L}_\mathbf{r}\cdot\mathbf{L}_{\mathbf{r}'}\Big]. \end{aligned} \tag{8.128}$$

The functional integral over $\mathbf{L}$ gives

$$\begin{aligned} \int \mathcal{D}[\mathbf{L}] \exp\bigg\{ -\frac{1}{2}\int_0^\beta d\tau \sum_{\mathbf{r},\mathbf{r}'} \chi^{-1}_{\perp\mathbf{r},\mathbf{r}'}\mathbf{L}_\mathbf{r}\cdot\mathbf{L}_{\mathbf{r}'} + i\int_0^\beta d\tau\sum_{\mathbf{r}}\mathbf{L}_\mathbf{r}\cdot(\mathbf{n}_\mathbf{r}\times\partial_\tau\mathbf{n}_\mathbf{r})\bigg\} \\ = \exp\bigg\{ -\frac{1}{2}\int_0^\beta d\tau \sum_{\mathbf{r},\mathbf{r}'} \chi_{\perp\mathbf{r},\mathbf{r}'}(\mathbf{n}_\mathbf{r}\times\partial_\tau\mathbf{n}_\mathbf{r})\cdot(\mathbf{n}_{\mathbf{r}'}\times\partial_\tau\mathbf{n}_{\mathbf{r}'})\bigg\}. \end{aligned} \tag{8.129}$$

Since the scalar product $\mathbf{L}_\mathbf{r}\cdot(\mathbf{n}_\mathbf{r}\times\partial_\tau\mathbf{n}_\mathbf{r})$ involves only the component of $\mathbf{L}_\mathbf{r}$ which is perpendicular to $\mathbf{n}_\mathbf{r}$, we can ignore the constraint $\mathbf{L}_\mathbf{r}\perp\mathbf{n}_\mathbf{r}$ when performing the integration over $\mathbf{L}_\mathbf{r}$. To second order in $\partial_\tau\mathbf{n}_\mathbf{r}$, we obtain the contribution to the action[35]

$$\frac{1}{2}\int_0^\beta d\tau\sum_\mathbf{r}\chi_\perp(\mathbf{n}_\mathbf{r}\times\partial_\tau\mathbf{n}_\mathbf{r})^2 = \frac{1}{2}\int_0^\beta d\tau\sum_\mathbf{r}\chi_\perp(\partial_\tau\mathbf{n}_\mathbf{r})^2 \tag{8.130}$$

with $\chi_\perp \equiv \chi_\perp(\mathbf{q}=0) = 1/4|J|d$ [Eq. (8.125)]. Taking the continuum limit, we finally obtain

$$\begin{aligned} Z = \int\mathcal{D}[\mathbf{n}]\,\delta(\mathbf{n}^2-1)\exp\{-\mathcal{S}[\mathbf{n}]\}, \\ \mathcal{S}[\mathbf{n}] = S'_B[\mathbf{n}] + \frac{\rho_s^0}{2}\int_0^\beta d\tau\int d^dr\Big[(\boldsymbol{\nabla}\mathbf{n})^2 + \frac{1}{c^2}(\partial_\tau\mathbf{n})^2\Big]. \end{aligned} \tag{8.131}$$

This action $\mathcal{S}[\mathbf{n}]$ should be supplemented with a ultraviolet momentum cutoff $\Lambda \ll 1$ below which the continuum approximation is valid (in practice one often takes a cutoff of the order of the inverse lattice spacing). Equation (8.131) corresponds to a quantum nonlinear sigma model with an additional Berry phase term $\mathcal{S}'_B$.[28] $\rho_s^0 = |J|S^2$ is the (bare) stiffness of the nonlinear sigma model and $c = (\rho_s^0/\chi_\perp)^{1/2} = 2\sqrt{d}|J|S$ the (bare) velocity of the spinwave modes.

One-dimensional antiferromagnet

Let us rewrite the Berry phase term $\mathcal{S}'_B$ in the following form

$$\mathcal{S}'_B = iS\sum_i(\omega[\mathbf{n}_{2i+1}]-\omega[\mathbf{n}_{2i}]) \simeq iS\sum_i\int_0^\beta d\tau\,(\mathbf{n}_{2i+1}-\mathbf{n}_{2i})\cdot\frac{\delta\omega[\mathbf{n}_{2i}]}{\delta\mathbf{n}_{2i}(\tau)}, \tag{8.132}$$

[35] $\int_0^\beta d\tau\sum_{\mathbf{r},\mathbf{r}'}\chi_{\perp\mathbf{r},\mathbf{r}'}(\mathbf{n}_\mathbf{r}\times\partial_\tau\mathbf{n}_\mathbf{r})\cdot(\mathbf{n}_{\mathbf{r}'}\times\partial_\tau\mathbf{n}_{\mathbf{r}'}) = \sum_q\chi_\perp(\mathbf{q})(\mathbf{n}\times\partial_\tau\mathbf{n})_{-q}\cdot(\mathbf{n}\times\partial_\tau\mathbf{n})_q \simeq \sum_q\chi_\perp(\mathbf{q}=0)(\mathbf{n}\times\partial_\tau\mathbf{n})_{-q}\cdot(\mathbf{n}\times\partial_\tau\mathbf{n})_q$ to second order in derivatives, hence Eq. (8.130).

assuming that n_i varies smoothly in space. Here $\mathbf{n}_i(\tau)$ stands for $\mathbf{n}_{x_i}(\tau)$ and $\omega[\mathbf{n}_i]$ denotes the solid angle defined by the closed trajectory of $\mathbf{n}_i(\tau)$ between times 0 and β (see Eq. (8.98)). Using $\delta\omega[\mathbf{n}]/\delta\mathbf{n}(\tau) = -\mathbf{n}(\tau)\times\partial_\tau\mathbf{n}(\tau)$ [Eq. (8.103)], we obtain

$$\begin{aligned} \mathcal{S}'_B &= -iS\int_0^\beta d\tau \sum_i (\mathbf{n}_{2i+1}-\mathbf{n}_{2i})\cdot(\mathbf{n}_{2i}\times\partial_\tau\mathbf{n}_{2i}) \\ &\simeq -\frac{i}{2}S\int_0^L dx\int_0^\beta d\tau\,\partial_x\mathbf{n}\cdot(\mathbf{n}\times\partial_\tau\mathbf{n}) \end{aligned} \tag{8.133}$$

in the continuum limit. This result can be rewritten as $\mathcal{S}'_B = i2\pi S\Theta$ where

$$\Theta[\mathbf{n}] = \frac{1}{4\pi}\int_0^L dx\int_0^\beta d\tau\,\mathbf{n}\cdot(\partial_x\mathbf{n}\times\partial_\tau\mathbf{n}) \tag{8.134}$$

is the topological winding number (or Pontryagin index) of the mapping $\mathbf{n}:\ [0,L]\times[0,\beta]\to S^2_{\rm int}$ with $S^2_{\rm int}$ the surface of the unit sphere, i.e. the mapping from two-dimensional spacetime to the unit sphere $S^2_{\rm int}$. If we assume that the field approaches a constant value on the boundary of its domain of definition, then the spacetime is topologically equivalent to the unit sphere and $\mathbf{n}$ realizes a mapping from $S^2_{\rm phys}$ to $S^2_{\rm int}$.[36] It turns out that $\Theta[\mathbf{n}]$ counts how many times spacetime has been wrapped around the sphere and must therefore be an integer.

To see this, let us introduce the two-dimensional coordinate $(x_1,x_2)\equiv(x,\tau)$ (simply denoted by x in the following):

$$\begin{aligned} \Theta[\mathbf{n}] &= \frac{1}{8\pi}\int d^2x\,\epsilon_{\mu\nu}\mathbf{n}\cdot(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}) \\ &= \frac{1}{8\pi}\int d^2x\,\epsilon_{\mu\nu}\epsilon_{abc}n^a\frac{\partial n^b}{\partial x_\mu}\frac{\partial n^c}{\partial x_\nu} \\ &= \frac{1}{8\pi}\int d^2x\,\epsilon_{\mu\nu}\epsilon_{abc}n^a\frac{\partial n^b}{\partial \xi_r}\frac{\partial \xi_r}{\partial x_\mu}\frac{\partial n^c}{\partial \xi_s}\frac{\partial \xi_s}{\partial x_\nu} \end{aligned} \tag{8.135}$$

(a sum over repeated indices is implied) where $\xi=(\xi_1,\xi_2)\equiv(\theta,\varphi)$ with θ and φ the polar and azimuthal angles corresponding to the unit vector $\mathbf{n}$. Now

$$d^2\xi = d^2x\,\epsilon_{\mu\nu}\frac{\partial\xi_1}{\partial x_\mu}\frac{\partial\xi_2}{\partial x_\nu}, \qquad \epsilon_{rs}d^2\xi = d^2x\,\epsilon_{\mu\nu}\frac{\partial\xi_r}{\partial x_\mu}\frac{\partial\xi_s}{\partial x_\nu} \tag{8.136}$$

so that

$$\Theta[\mathbf{n}] = \frac{1}{8\pi}\int d^2\xi\,\epsilon_{abc}\epsilon_{rs}n^a\frac{\partial n^b}{\partial\xi_r}\frac{\partial n^c}{\partial\xi_s}. \tag{8.137}$$

Consider now an element $d\mathbf{S}_{\rm int}=(dS^1_{\rm int},dS^2_{\rm int},dS^3_{\rm int})$ of the surface of the unit sphere $S^2_{\rm int}$. Noting that

$$dn^b dn^c = \epsilon_{rs}\frac{\partial n^b}{\partial\xi_r}\frac{\partial n^c}{\partial\xi_s}d^2\xi, \tag{8.138}$$

[36] Here we distinguish between the unit sphere in physical space ($S^2_{\rm phys}$) and and the one in internal space ($S^2_{\rm int}$).

one has

$$dS^a_{\rm int} = \frac{1}{2}\epsilon_{abc}\epsilon_{rs}\frac{\partial n^b}{\partial \xi_r}\frac{\partial n^c}{\partial \xi_s}d^2\xi. \tag{8.139}$$

Comparing (8.137) and (8.139), we obtain

$$\Theta[\mathbf{n}] = \frac{1}{4\pi}\int dS^a_{\rm int} n^a = \frac{1}{4\pi}\int dS_{\rm int}. \tag{8.140}$$

Since the unit sphere $S^2_{\rm int}$ has area 4π, we clearly see that $\Theta[\mathbf{n}]$ gives the number of times the internal sphere is traversed as we span the coordinate space as compactified into $S^2_{\rm phys}$.

We are therefore led to distinguish integer and half-integer spins. In the former case, $e^{-\mathcal{S}'_B} = 1$ and the effective action $\mathcal{S}[\mathbf{n}]$ reduces to that of the one-dimensional quantum nonlinear model. At zero temperature, the latter is equivalent to the classical two-dimensional nonlinear model. The correlation length is finite,[37]

$$\xi \sim \Lambda^{-1} e^{\frac{2\pi\rho^0_s}{c}} \simeq \Lambda^{-1} e^{\pi S}. \tag{8.141}$$

The finite correlation length implies a nonzero gap in the spectrum (the so-called Haldane gap): $\Delta = c/\xi \sim c\Lambda e^{-\pi S}$.

For half-integer spins, the Berry phase term $\mathcal{S}'_B = i2\pi S\Theta$ cannot be ignored. The action $\mathcal{S}[\mathbf{n}]$ [Eq. (8.131)] is difficult to study but shows that the low-energy behavior of all half-integer spin chains is the same. From the Lieb, Schultz and Mattis theorem [22] we know that the spectrum is gapless in the thermodynamic limit. There are powerful methods to study half-integer spin chains. For instance, in the case $S = \frac{1}{2}$ the spectrum is given exactly by the Bethe ansatz solution. The low-energy behavior can be obtained from "bosonization" (a low-energy effective approach). The spin-$\frac{1}{2}$ Heisenberg chain belongs to a class of one-dimensional fluids known as "Luttinger liquids" characterized by a gapless spectrum and power-law decaying correlation functions.[38]

This has led to the conjecture that half-integer-spin chains, regardless of the value of S, are gapless while integer-spin chains are gapped (Haldane's conjecture) [14]. No counter-examples have invalidated this conjecture so far.

Two-dimensional antiferromagnet

Let us first ignore the Berry phase term. At zero temperature the two-dimensional quantum nonlinear sigma model is equivalent to the classical three-dimensional nonlinear sigma model. There is a quantum phase transition between an ordered state with spontaneously broken spin-rotation invariance and a disordered paramagnetic state. The renormalization-group approach to one-loop order predicts the transition to occur at the critical value $\tilde g_c = 4\pi^2$ of the dimensionless coupling constant $\tilde g = gc\Lambda$ of the nonlinear sigma model (Sec. 10.7.2). In the case of interest here, $g = 1/\rho^0_s$ and $\tilde g = 2\sqrt{2}\Lambda/S$. For large S, $\tilde g \ll \tilde g_c$ and the ground state exhibits long-range antiferromagnetic order. For S of order one, it is difficult to conclude since the ultraviolet

[37] Eq. (8.141) is obtained from (10.306) with $1/g = \rho^0_s/c$.

[38] Besides spin chains, Luttinger liquids include one-dimensional interacting bosons and fermions (see chapters 13 and 15).

momentum cutoff Λ is not precisely known. Nevertheless it is known from numerical calculations that even for $S = \frac{1}{2}$, the ground state of the Heisenberg on the square lattice has antiferromagnetic long-range order [2]. At finite temperatures and $g < g_c$, the quantum nonlinear sigma model is disordered by thermal fluctuations. This low-energy behavior is then well described by a classical nonlinear sigma model but with a stiffness renormalized by quantum fluctuations (see Sec. 12.3). The antiferromagnetic correlation length diverges at low temperatures as

$$\xi \sim \frac{c}{T} e^{\frac{2\pi\rho_s}{T}}, \tag{8.142}$$

where ρ_s is the zero-temperature stiffness ($\rho_s < \rho_s^0$).

Consider now the Berry phase term $\mathcal{S}'_B$ for a smooth space-time configuration of $\mathbf{n}(\mathbf{r}, \tau)$ and evaluate (8.127) row by row on the square lattice. Apart from an oscillating sign (which we momentarily discard), for each row we recover the one-dimensional result, namely a result quantized in integer multiples of $2\pi S$. Since $\mathbf{n}$ varies slowly in space, the contribution of the rows must change smoothly as we move to row to row. This is compatible with the quantization only if each row yields precisely the same integer. Taking into account the oscillating sign, we conclude that $\mathcal{S}'_B$ must vanish. Thus a nonzero Berry phase can only be due to singular configurations. For a 3-component vector field, the only topologically stable possibility is the so-called hedgehog singularity. The hedgehog events do not play any role in the Néel phase ($g < g_c$) so that all previous conclusions regarding the two-dimensional Heisenberg model on the square lattice are valid even when $\mathcal{S}'_B$ is considered. But for $g > g_c$ they proliferate and lead to a spontaneously broken lattice symmetry unless S is an even integer (in which case the Berry phase term $\mathcal{S}'_B$ has no effect). The ground state, referred to as a valence bond solid (or a spin-Peierls state), can be characterized by the quantity $P_{\mathbf{r},\mathbf{r}'} = \langle \hat{\mathbf{S}}_\mathbf{r} \cdot \hat{\mathbf{S}}_{\mathbf{r}'} \rangle$, which is a scalar under spin rotations so that a nonzero value does not break spin-rotation invariance. In the valence bond solid, $P_{\mathbf{r},\mathbf{r}'}$ is such that it breaks the lattice translation invariance.

Since the Néel state and the valence bond solid break different symmetries (spin-rotation invariance and lattice translation symmetry, respectively), Landau's theory of phase transitions predicts the transition to be first order. The Berry phase term allows for a different scenario where the theory contains an emergent gauge field and deconfined degrees of freedom (spinons) at the quantum critical point, whereas both phases are characterized by conventional "confining" order parameters.[39]

8.5 Schwinger-boson mean-field theory

The symmetric phases of the Heisenberg model are easier to describe using a representation in which the rotational invariance remains manifest. In this section, we discuss a mean-field solution of the antiferromagnetic Heisenberg model where the spin operators are represented by Schwinger bosons.

[39] We refer to chapter 19 of Ref. [10] for a further discussion of the role of the Berry phase term in two-dimensional antiferromagnets and a discussion of deconfined criticality.

8.5.1 Schwinger bosons

A spin-S operator can be represented by 2 boson operators $\hat b_\uparrow$ and $\hat b_\downarrow$,

$$\hat{\mathbf{S}} = \frac{1}{2}\hat b^\dagger \boldsymbol{\sigma} \hat b, \qquad \hat b = \begin{pmatrix} \hat b_\uparrow \\ \hat b_\downarrow \end{pmatrix} \tag{8.143}$$

($\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ stands for the Pauli matrices), satisfying the constraint

$$\hat b^\dagger \hat b = \sum_\sigma \hat b^\dagger_\sigma \hat b_\sigma = 2S. \tag{8.144}$$

More explicitly, the spin operator is expressed as

$$\begin{aligned} \hat S^+ &= \hat S^x + i\hat S^y = \hat b^\dagger_\uparrow \hat b_\downarrow, \\ \hat S^- &= \hat S^x - i\hat S^y = \hat b^\dagger_\downarrow \hat b_\uparrow, \\ \hat S^z &= \frac{1}{2}(\hat b^\dagger_\uparrow \hat b_\uparrow - \hat b^\dagger_\downarrow \hat b_\downarrow). \end{aligned} \tag{8.145}$$

It is easy to verify that the relation $\hat{\mathbf{S}}^2 = S(S+1)$ and the spin commutation relations $\hat{\mathbf{S}} \times \hat{\mathbf{S}} = i\hat{\mathbf{S}}$ follow from the bosonic commutation relations $[\hat b_\sigma, \hat b^\dagger_{\sigma'}] = \delta_{\sigma,\sigma'}$ and $[\hat b_\sigma, \hat b_{\sigma'}] = [\hat b^\dagger_\sigma, \hat b^\dagger_{\sigma'}] = 0$.

8.5.2 Mean-field theory of the antiferromagnetic Heisenberg model

Defining Schwinger bosons $\hat b_{\mathbf{r}}, \hat b^\dagger_{\mathbf{r}}$ at each site of the lattice,[40] we can rewrite the Heisenberg model as

$$\hat H = -\frac{J}{2} \sum_{\langle \mathbf{r},\mathbf{r}'\rangle} \left(2S^2 - \hat{\mathcal{A}}^\dagger_{\mathbf{r}\mathbf{r}'} \hat{\mathcal{A}}_{\mathbf{r}\mathbf{r}'}\right), \tag{8.146}$$

with $\hat{\mathcal{A}}_{\mathbf{r}\mathbf{r}'} = \hat b_{\mathbf{r}\uparrow}\hat b_{\mathbf{r}'\downarrow} - \hat b_{\mathbf{r}\downarrow}\hat b_{\mathbf{r}'\uparrow}$. In the case $S = 1/2$, $\hat{\mathcal{A}}^\dagger_{\mathbf{r}\mathbf{r}'}$ creates a singlet on the bond $(\mathbf{r},\mathbf{r}')$: $\hat{\mathcal{A}}^\dagger_{\mathbf{r}\mathbf{r}'}|\text{vac}\rangle = (\hat b^\dagger_{\mathbf{r}\uparrow}\hat b^\dagger_{\mathbf{r}'\downarrow} - \hat b^\dagger_{\mathbf{r}\downarrow}\hat b^\dagger_{\mathbf{r}'\uparrow})|\text{vac}\rangle$ (with $|\text{vac}\rangle$ the vacuum of Schwinger bosons). Thus we expect that short-range antiferromagnetic order will be characterized by a nonzero value of $\langle \hat{\mathcal{A}}_{\mathbf{r}\mathbf{r}'}\rangle$, which suggests to decouple the product $\hat{\mathcal{A}}^\dagger\hat{\mathcal{A}}$ in (8.146) following the usual mean-field procedure.

At this point, it is convenient to make a spin rotation of angle π about the y axis on one of the two sublattices (e.g. sublattice B). This is equivalent to the change of variables $\hat b_\downarrow \to \hat b_\uparrow$ and $\hat b_\uparrow \to -\hat b_\downarrow$,[41] which is a canonical transformation preserving the bosonic commutation relations and the constraint (8.144). This leaves the form (8.146) of the Hamiltonian unchanged but the bond operator is now defined by

$$\hat{\mathcal{A}}_{\mathbf{r}\mathbf{r}'} = \hat b_{\mathbf{r}\uparrow}\hat b_{\mathbf{r}'\uparrow} + \hat b_{\mathbf{r}\downarrow}\hat b_{\mathbf{r}'\downarrow}. \tag{8.147}$$

We solve the Hamiltonian in a mean-field approximation which will be justified in section 8.5.3. On the one hand we impose the constraint (8.144) only on average,

[40] Schwinger bosons on different sites commute.

[41] From Eqs. (8.145), we verity that the transformation of the boson operators implies $\hat S^x \to -\hat S^x$, $\hat S^y \to \hat S^y$ and $\hat S^z \to -\hat S^z$.

$\langle \hat{b}^\dagger_{\mathbf{r}} \hat{b}_{\mathbf{r}} \rangle = 2S$. This is done by introducing a Lagrange multiplier $\lambda \equiv \lambda_{\mathbf{r}}$. On the other hand we decouple the quartic bosonic term of the Hamiltonian: $\hat{\mathcal{A}}^\dagger \hat{\mathcal{A}} \to \langle \hat{\mathcal{A}}^\dagger \rangle \hat{\mathcal{A}} + \hat{\mathcal{A}}^\dagger \langle \hat{\mathcal{A}} \rangle - \langle \hat{\mathcal{A}}^\dagger \rangle \langle \hat{\mathcal{A}} \rangle$. This yields the Hamiltonian

$$\hat{H}_{\rm MF} = E_0 + \lambda \sum_{\mathbf{r}} \hat{b}^\dagger_{\mathbf{r}} \hat{b}_{\mathbf{r}} + \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} \big(Q^* \hat{\mathcal{A}}_{\mathbf{r}\mathbf{r}'} + \text{h.c.} \big), \tag{8.148}$$

where

$$Q = \frac{J}{2} \langle \hat{\mathcal{A}}_{\mathbf{r}\mathbf{r}'} \rangle, \quad Q^* = \frac{J}{2} \langle \hat{\mathcal{A}}^\dagger_{\mathbf{r}\mathbf{r}'} \rangle, \quad E_0 = -\lambda 2SN - \frac{z}{J} |Q|^2 N \tag{8.149}$$

($z = 2d$ is the number of nearest neighbors). In Fourier space,

$$\hat{H}_{\rm MF} = E_0 + \lambda \sum_{\mathbf{q}} \hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}} + \frac{z}{2} \sum_{\mathbf{q},\sigma} \gamma_{\mathbf{q}} Q (\hat{b}_{\mathbf{q}\sigma} \hat{b}_{-\mathbf{q}\sigma} + \text{h.c.}), \tag{8.150}$$

where, with no loss of generality, we now taken Q real. $\hat{H}_{\rm MF}$ can be diagonalized by the (canonical) Bogoliubov transformation

$$\hat{b}_{\mathbf{q}\sigma} = \cosh(\theta_{\mathbf{q}}) \hat{\alpha}_{\mathbf{q}\sigma} + \sinh(\theta_{\mathbf{q}}) \hat{\alpha}^\dagger_{-\mathbf{q}\sigma}, \tag{8.151}$$

where we assume $\theta_{\mathbf{q}} = \theta_{-\mathbf{q}}$. The Hamiltonian

$$\begin{aligned} \hat{H}_{\rm MF} = E_0 + \sum_{\mathbf{q},\sigma} \Big\{ & \frac{1}{2} [\lambda \sinh(2\theta_q) + zQ\gamma_{\mathbf{q}} \cosh(2\theta_{\mathbf{q}})] (\hat{\alpha}_{\mathbf{q}\sigma} \hat{\alpha}_{-\mathbf{q}\sigma} + \text{h.c.}) \\ & + [\lambda \cosh(2\theta_{\mathbf{q}}) + zQ\gamma_{\mathbf{q}} \sinh(2\theta_q)] \hat{\alpha}^\dagger_{\mathbf{q}\sigma} \hat{\alpha}_{\mathbf{q}\sigma} + \lambda \sinh^2(\theta_{\mathbf{q}}) + \frac{z}{2} Q \gamma_{\mathbf{q}} \sinh(2\theta_q) \Big\} \end{aligned} \tag{8.152}$$

takes a simple form if we choose $\tanh(2\theta_{\mathbf{q}}) = -zQ\gamma_{\mathbf{q}}/\lambda$,

$$\hat{H}_{\rm MF} = -N \Big[\lambda(2S+1) + \frac{z}{J} Q^2 \Big] + \sum_{\mathbf{q},\sigma} \omega_{\mathbf{q}} \Big(\hat{\alpha}^\dagger_{\mathbf{q}\sigma} \hat{\alpha}_{\mathbf{q}\sigma} + \frac{1}{2} \Big), \tag{8.153}$$

where

$$\omega_{\mathbf{q}} = (\lambda^2 - z^2 Q^2 \gamma_{\mathbf{q}}^2)^{1/2} \tag{8.154}$$

is the dispersion of the bosons $\hat{\alpha}$. The spectrum has a gap $\Delta = (\lambda^2 - z^2 Q^2)^{1/2}$ and the minimum excitation energy is reached for $\mathbf{q} = 0$ and $\mathbf{q} = \mathbf{Q}$. $\omega_{\mathbf{q}} \simeq (\Delta^2 + c^2 \mathbf{q}^2)^{1/2}$ for $\mathbf{q} \to 0$ and $\omega_{\mathbf{q}} \simeq (\Delta^2 + c^2 (\mathbf{q} - \mathbf{Q})^2)^{1/2}$ for $\mathbf{q} \to \mathbf{Q}$ where $c = \sqrt{2z}|Q|$. If $\lambda/z|Q| \to 1$, the gap vanishes and we expect Bose-Einstein condensation, i.e. antiferromagnetic long-range order (see the discussion of the two-dimensional antiferromagnet below). A nonzero gap Δ implies a finite antiferromagnetic correlation length ξ. The boson propagator decays exponentially in space with a characteristic length c/Δ.[42] The spin-spin correlation function is a two-particle correlation function and should therefore decay as the square of the boson propagator, i.e. $\xi = c/2\Delta$.[43]

[42] This follows from $\omega_{\mathbf{q}} \simeq (\Delta^2 + c^2 \mathbf{q}^2)^{1/2}$ for $\mathbf{q} \to 0$.
[43] For an explicit calculation of the spin-spin correlation function see [1, 24].

Since the Hamiltonian reduces to a sum of decoupled harmonic oscillators, the free energy is given by

$$F = -N\Big[\lambda(2S+1) + \frac{z}{J}Q^2\Big] + \frac{2}{\beta}\sum_{\mathbf{q}} \ln\Big[2\sinh\Big(\frac{\beta}{2}\omega_{\mathbf{q}}\Big)\Big]. \tag{8.155}$$

The values of λ and Q are obtained from $\langle \hat b^\dagger_{\mathbf{r}} \hat b_{\mathbf{r}}\rangle = 2S$ and $Q = \langle \hat{\mathcal{A}}_{\mathbf{r}\mathbf{r}'}\rangle$ or, equivalently, by minimizing the free energy, $\partial F/\partial\lambda = \partial F/\partial Q = 0$:

$$\begin{aligned} S + \frac{1}{2} &= \int_{\mathbf{q}} \left(n_{\mathbf{q}} + \frac{1}{2}\right)\frac{\lambda}{\omega_{\mathbf{q}}}, \\ \frac{Q}{|J|} &= \int_{\mathbf{q}} \left(n_{\mathbf{q}} + \frac{1}{2}\right) zQ\frac{\gamma_{\mathbf{q}}^2}{\omega_{\mathbf{q}}}, \end{aligned} \tag{8.156}$$

where $n_{\mathbf{q}} = n_B(\omega_{\mathbf{q}})$.

One-dimensional antiferromagnet

In one dimension and for $T = 0$, the first mean-field equation (8.156) gives

$$S + \frac{1}{2} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{dq}{2\pi}\frac{1}{\sqrt{1-\eta^2\cos(q)^2}} = \frac{1}{\pi}K(\eta), \tag{8.157}$$

where $\eta = 2|Q|/\lambda$ and K is the complete elliptic integral of the first kind.[44] Equation (8.157) determines the correlation length $\xi = \eta/2\sqrt{1-\eta^2}$ as a function of S.

In the large-S limit, using $K(\eta) \simeq \ln(4/\sqrt{1-\eta^2})$ for $\eta \to 1$, one finds

$$\xi \simeq \frac{e^{\pi S}}{8} \qquad (S \gg 1), \tag{8.158}$$

which agrees with the result obtained from the nonlinear sigma model for integer S [Eq. (8.141)]. The Schwinger-boson mean-field theory is not adequate for half-integer spin chains since it fails to obtain a gapless spectrum. Subtracting the first mean-field equation from the second one, and using $\lambda \simeq z|Q|$ when $S \gg 1$, one obtains

$$c - 2|J|\left(S + \frac{1}{2}\right) = -|J|\int_q \frac{\lambda}{\omega_q}\sin(q)^2 \simeq -\frac{2}{\pi}|J|, \tag{8.159}$$

and in turn

$$c \simeq 2|J|\left(S + \frac{1}{2} - \frac{1}{\pi}\right) \simeq 2|J|S, \tag{8.160}$$

in agreement with the spinwave result.[45]

[44] $K(k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$.

[45] Let us recall that the spinwave analysis is not justified in one dimension since it assumes long-range order.

At finite temperature, when $T \gg \Delta$ (i.e. $\xi \gg c/T$) we can use $n_q \simeq T/\omega_q$ so that the mean-field equations become

$$\begin{aligned} S + \frac{1}{2} &= \int_q \frac{\lambda T}{\omega_q^2} = \frac{T}{\Delta}, \\ \frac{1}{|J|} &= 2T \int_q \frac{\cos^2 q}{\omega_q^2} = \frac{2T}{c^2} \left(\frac{\sqrt{\Delta^2 + c^2}}{\Delta} - 1 \right). \end{aligned} \tag{8.161}$$

In the large-S limit, one finds $\Delta \simeq T/S$ and $c \simeq 2|J|S$, i.e. a correlation length $\xi \sim |J|S^2/T \gg c/T$ in agreement with the conclusions drawn from the spinwave expansion in section 8.3.2.

Two-dimensional antiferromagnet

For $d = 2$ and $T = 0$, the first mean-field equation gives

$$S + \frac{1}{2} = \frac{1}{2} \int_{-1}^{1} d\gamma \frac{\rho(\gamma)}{\sqrt{1 - \eta^2 \gamma^2}}, \tag{8.162}$$

where $\eta = 4|Q|/\lambda$ and

$$\rho(\gamma) = \int_{\mathbf{q}} \delta(\gamma - \gamma_{\mathbf{q}}) = \frac{2}{\pi^2} K(\sqrt{1 - \gamma^2}) \tag{8.163}$$

is the density of states associated with the dispersion $\gamma_{\mathbf{q}}$. The rhs in (8.162) is maximum for $\eta = 1$ and then equal to 0.6966. For $S \geq 0.1966$, equation (8.162) has therefore no solution and we must allow for Bose-Einstein condensation in the states $\mathbf{q} = 0$ and $\mathbf{q} = (\pi, \pi)$ (the two momenta where $\omega_{\mathbf{q}} = 0$ when $\eta = 1$).

More precisely we consider a condensate defined by

$$\langle \hat{b}_{\mathbf{r}\sigma} \rangle = \frac{1}{2} [1 + \sigma(-1)^{\mathbf{r}}] \sqrt{n_0} e^{i\theta}, \tag{8.164}$$

with $n_0 = |\langle b_{\mathbf{r}\sigma} \rangle|^2$ the condensate density and θ an arbitrary phase. The condensate corresponds to a staggered magnetization along the z axis [Eqs. (8.145)],

$$\langle \hat{S}^z_{\mathbf{r}} \rangle = \frac{1}{2} \left[|\langle \hat{b}_{\mathbf{r}\uparrow} \rangle|^2 - |\langle \hat{b}_{\mathbf{r}\downarrow} \rangle|^2 \right] = \frac{n_0}{2} (-1)^{\mathbf{r}}. \tag{8.165}$$

In the absence of quantum fluctuations, all bosons would be in the condensate, $\sum_\sigma |\langle \hat{b}_{\mathbf{r}\sigma} \rangle|^2 = 2S$, which would give a magnetization $m_0 = n_0/2 = S$ as in the classical Néel state. After the spin-rotation of angle π about the y axis for the B sublattice, the condensate is defined by $\langle \hat{b}_{\mathbf{r}\sigma} \rangle = \sqrt{n_0} e^{i\theta} \delta_{\sigma,\uparrow}$. Thus the condensate yields the additional contribution

$$N n_0 \Big[\lambda + \frac{z}{2} (Q^* e^{2i\theta} + \text{c.c.}) \Big] \tag{8.166}$$

to the mean-field Hamiltonian (8.148), where we now consider a complex order parameter Q. If we take $\theta = 0$, the mean-field free energy is minimized for a real and negative order parameter Q. The mean-field equations become

$$\begin{aligned} S+\frac{1}{2} &= \frac{n_0}{2} + \int_{\mathbf{q}} \left(n_{\mathbf{q}} + \frac{1}{2} \right) \frac{\lambda}{\omega_{\mathbf{q}}}, \\ \frac{Q}{|J|} &= -\frac{n_0}{2} + \int_{\mathbf{q}} \left(n_{\mathbf{q}} + \frac{1}{2} \right) 4Q \frac{\gamma_{\mathbf{q}}^2}{\omega_{\mathbf{q}}}, \end{aligned} \tag{8.167}$$

where the boson dispersion $\omega_{\mathbf{q}} = \lambda(1-\gamma_{\mathbf{q}}^2)^{1/2}$ vanishes for $\mathbf{q}=0$ and $\mathbf{q}=(\pi,\pi)$. The first equation gives the staggered magnetization in the ground state,

$$m_0 = \frac{n_0}{2} = S + \frac{1}{2} - \frac{1}{2} \int_{\mathbf{q}} \frac{1}{1-\gamma_{\mathbf{q}}^2} \simeq S - 0.1966, \tag{8.168}$$

and reproduces the result obtained from the spinwave expansion (Sec. 8.3.2). From (8.167) one also obtains

$$\frac{|Q|}{|J|} = S + \frac{1}{2} - \frac{1}{2} \int_{\mathbf{q}} \sqrt{1-\gamma_{\mathbf{q}}^2}, \tag{8.169}$$

i.e.

$$\begin{aligned} \frac{c}{2\sqrt{2}|J|S} &= 1 + \frac{1}{2S} \Big[1 - \frac{2}{\pi^2} \int_{-1}^{1} d\gamma \sqrt{1-\gamma^2} K(\sqrt{1-\gamma^2}) \Big] \\ &\simeq 1 + \frac{0.07897}{S}. \end{aligned} \tag{8.170}$$

The last term in (8.170) corresponds to a correction to the leading-order spinwave velocity $c = 2\sqrt{2}|J|S$ (Sec. 8.3.2).

At finite temperatures, the integral

$$\int_{\mathbf{q}} \frac{n_{\mathbf{q}}}{\omega_{\mathbf{q}}} \sim T \int_{\mathbf{q}} \frac{1}{\omega_{\mathbf{q}}^2} \tag{8.171}$$

is logarithmically divergent if the boson dispersion is gapless: $\omega_{\mathbf{q}} \simeq c|\mathbf{q}|$ for $\mathbf{q}\to 0$ and $\omega_{\mathbf{q}} \simeq c|\mathbf{q}-\mathbf{Q}|$ for $\mathbf{q}\to\mathbf{Q}=(\pi,\pi)$. In agreement with the Mermin-Wagner theorem, Bose-Einstein condensation is not possible at finite temperatures in two dimensions. Thus we must have a vanishing condensate density, $n_0=0$, and a gapped spectrum, $\eta<1$. Let us rewrite the mean-field equations as

$$\begin{aligned} S+\frac{1}{2} &= \frac{1}{2} \int_{-1}^{1} d\gamma \rho(\gamma) (1-\eta^2\gamma^2)^{-1/2} \operatorname{cotanh}\left(\frac{\lambda}{2T} \sqrt{1-\eta^2\gamma^2} \right), \\ S+\frac{1}{2} - \frac{\lambda\eta^2}{4|J|} &= \frac{1}{2} \int_{-1}^{1} d\gamma \rho(\gamma) (1-\eta^2\gamma^2)^{1/2} \operatorname{cotanh}\left(\frac{\lambda}{2T} \sqrt{1-\eta^2\gamma^2} \right) \end{aligned} \tag{8.172}$$

(the second one is obtained by combining the two equations of (8.167)). Following Ref. [24], we rewrite the first equation as

$$\begin{aligned} S+\frac{1}{2} &= \frac{2T}{\pi\lambda} \left[\frac{1}{2\eta} \ln\left(\frac{1+\eta}{1-\eta} \right) - \ln\left(\frac{2\lambda}{T} \right) \right] \\ &\quad + \frac{1}{2} \int_{-1}^{1} d\gamma \rho(\gamma) (1-\eta^2\gamma^2)^{-1/2} + \mathcal{O}(T^3), \end{aligned} \tag{8.173}$$

which gives (noting that $\eta \simeq 1$ at low temperatures)

$$\sqrt{1-\eta} \simeq \frac{T}{\lambda\sqrt{2}} \exp\left(-\frac{\pi m_0 \lambda}{2T}\right), \tag{8.174}$$

where m_0 is the ground-state magnetization [Eq. (8.168)]. The second equation (8.172) gives [24]

$$S + \frac{1}{2} - \frac{\lambda\eta^2}{4|J|} = \frac{1}{2}\int_{-1}^{1} d\gamma \rho(\gamma)(1-\eta^2\gamma^2)^{1/2} + \mathcal{O}(T^3) \tag{8.175}$$

and in turn

$$\frac{\lambda\eta^2}{4|J|} \simeq Z_c S, \tag{8.176}$$

where Z_c is defined by the rhs of (8.170). We therefore obtain $\lambda \simeq 4|J|Z_c S$,

$$\sqrt{1-\eta} = \frac{T}{4\sqrt{2}|J|Z_c S} \exp\left(-\frac{2\pi m_0 |J| Z_c S}{T}\right) \tag{8.177}$$

and

$$\xi = \frac{\eta}{2\sqrt{2}\sqrt{1-\eta^2}} \simeq \frac{c}{2T} \exp\left(\frac{2\pi m_0 |J| Z_c S}{T}\right), \tag{8.178}$$

which reproduces the temperature-dependence of the correlation length obtained from the quantum nonlinear sigma model (Sec. 8.4.3). By analogy with (8.142), we deduce that the stiffness takes the form

$$\rho_s = m_0 Z_c S |J| \tag{8.179}$$

in the zero-temperature ordered ground state. We recover $\rho_s \simeq \rho_s^{(0)} = |J|S^2$ in the large-S limit.

8.5.3 SU($\mathcal{N}$) Heisenberg model

In this section, we show that the Schwinger-boson mean-field theory can be put on a firm basis by extending the spin symmetry of the Heisenberg model from SU(2) to a SU($\mathcal{N}$). This is done by considering a $\mathcal{N}$-flavor operator $\hat{b}_{\mathbf{r}m}$ ($m = 1 \cdots \mathcal{N}$) and redefining the bond operator (8.147) as

$$\hat{\mathcal{A}}_{\mathbf{r}\mathbf{r}'} = \sum_{m=1}^{\mathcal{N}} \hat{b}_{\mathbf{r}m}\hat{b}_{\mathbf{r}'m} \quad \text{with} \quad \sum_{m=1}^{\mathcal{N}} \hat{b}^\dagger_{\mathbf{r}m}\hat{b}_{\mathbf{r}m} = \mathcal{N}S. \tag{8.180}$$

The Hamiltonian (8.146) becomes

$$\hat{H} = -\frac{J}{\mathcal{N}} \sum_{\langle \mathbf{r},\mathbf{r}'\rangle} \left(\mathcal{N}S^2 - \hat{\mathcal{A}}^\dagger_{\mathbf{r}\mathbf{r}'}\hat{\mathcal{A}}_{\mathbf{r}\mathbf{r}'}\right) \tag{8.181}$$

after a proper rescaling[46] of the exchange interaction J, and is invariant in a SU($\mathcal{N}$) rotation of the boson field $\hat{b}_{\mathbf{r}} = (\hat{b}_{\mathbf{r}1}, \cdots, \hat{b}_{\mathbf{r}\mathcal{N}})$.[47]

[46] This rescaling is necessary to have a well-defined large-$\mathcal{N}$ limit.

[47] The Hamiltonian (8.181) is invariant under suitably defined SU($\mathcal{N}$) "rotations". See chapter 16 of Ref. [1] for a discussion.

We can write the partition function as a functional integral

$$Z = \int \mathcal{D}[b^*, b, \lambda]\, e^{-\int_0^\beta d\tau \left\{ \sum_{\mathbf{r},m} \left[b^*_{\mathbf{r}m} \partial_\tau b_{\mathbf{r}m} + i\lambda_{\mathbf{r}} (b^*_{\mathbf{r}m} b_{\mathbf{r}m} - S) \right] + H[b^*, b] \right\}} \tag{8.182}$$

over a complex bosonic field $b_{\mathbf{r}}$ and a Lagrange multiplier $\lambda_{\mathbf{r}}$ which enforces the constraint $\hat{b}^\dagger_{\mathbf{r}} \hat{b}_{\mathbf{r}} = \mathcal{N} S$. We now decouple the Hamiltonian by means of a Hubbard-Stratonovich transformation,

$$e^{-\int_0^\beta d\tau\, H[b^\dagger, b]} = \int \mathcal{D}[Q^*, Q]\, e^{-\int_0^\beta d\tau \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} \left[\frac{\mathcal{N}}{|J|} |Q_{\mathbf{r}\mathbf{r}'}|^2 + Q^*_{\mathbf{r}\mathbf{r}'} \mathcal{A}_{\mathbf{r}\mathbf{r}'} + Q_{\mathbf{r}\mathbf{r}'} \mathcal{A}^*_{\mathbf{r}\mathbf{r}'} \right]}, \tag{8.183}$$

where the auxiliary field $Q_{\mathbf{r}\mathbf{r}'}$ is defined for each pair $(\mathbf{r}, \mathbf{r}')$ of nearest neighbors. The action is now a function of b^*, b, λ and Q^*, Q. Integrating out the boson field, we obtain

$$\int \mathcal{D}[b^*, b]\, e^{-\int_0^\beta d\tau \sum_m \left\{ \sum_{\mathbf{r}} b^*_{\mathbf{r}m} (\partial_\tau + i\lambda_{\mathbf{r}}) b_{\mathbf{r}m} + \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} (Q^*_{\mathbf{r}\mathbf{r}'} b_{\mathbf{r}m} b_{\mathbf{r}'m} + \text{c.c.}) \right\}} = (\det \mathcal{G})^{\mathcal{N}/2}, \tag{8.184}$$

where the boson propagator $\mathcal{G} \equiv \mathcal{G}[Q^*, Q, \lambda]$ is defined by

$$\mathcal{G}^{-1}_{\mathbf{r}\mathbf{r}'}(\tau - \tau') = \begin{pmatrix} (\partial_\tau + i\lambda_{\mathbf{r}}) \delta_{\mathbf{r},\mathbf{r}'} & Q_{\mathbf{r}\mathbf{r}'} \\ Q^*_{\mathbf{r}\mathbf{r}'} & (-\partial_\tau + i\lambda_{\mathbf{r}}) \delta_{\mathbf{r},\mathbf{r}'} \end{pmatrix} \delta(\tau - \tau'), \tag{8.185}$$

and $Q^{(*)}_{\mathbf{r}\mathbf{r}'}$ vanishes if $\mathbf{r}$ and $\mathbf{r}'$ are not nearest neighbors. Putting everything together, we obtain

$$Z = \int \mathcal{D}[Q^*, Q, \lambda]\, e^{-\mathcal{N} \mathcal{S}[\mathcal{Q}^*, \mathcal{Q}, \lambda]}, \tag{8.186}$$

where

$$\mathcal{S}[Q^*, Q, \lambda] = \int_0^\beta d\tau \left[\frac{1}{|J|} \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} |Q_{\mathbf{r}\mathbf{r}'}|^2 - iS \sum_{\mathbf{r}} \lambda_{\mathbf{r}} \right] - \frac{1}{2} \mathrm{Tr} \ln \mathcal{G}[Q^*, Q, \lambda] \tag{8.187}$$

is independent of $\mathcal{N}$. In the limit $\mathcal{N} \to \infty$, the saddle-point approximation becomes exact (see Sec. 1.7). The saddle-point equations are

$$\begin{aligned} 0 &= \frac{\delta \ln Z}{\delta \lambda_{\mathbf{r}}(\tau)} = -i \sum_m \langle b^*_{\mathbf{r}m} b_{\mathbf{r}m} \rangle + i\mathcal{N} S, \\ 0 &= \frac{\delta \ln Z}{\delta Q^*_{\mathbf{r}\mathbf{r}'}(\tau)} = -\frac{\mathcal{N}}{|J|} Q_{\mathbf{r}\mathbf{r}'} - \langle \mathcal{A}_{\mathbf{r}\mathbf{r}'} \rangle, \\ 0 &= \frac{\delta \ln Z}{\delta Q_{\mathbf{r}\mathbf{r}'}(\tau)} = -\frac{\mathcal{N}}{|J|} Q^*_{\mathbf{r}\mathbf{r}'} - \langle \mathcal{A}^*_{\mathbf{r}\mathbf{r}'} \rangle. \end{aligned} \tag{8.188}$$

Assuming uniform and static fields: $\lambda_{\mathbf{r}}(\tau) = i\lambda$, $Q^{(*)}_{\mathbf{r}\mathbf{r}'}(\tau) = Q^{(*)}$, we recover the mean-field equations studied in section 8.5.2 for $\mathcal{N} = 2$.

The mean-field theory is exact only in the limit $\mathcal{N} \to \infty$. For $\mathcal{N} < \infty$, some nonphysical states which violate the local constraint $\hat{b}^\dagger_{\mathbf{r}} \hat{b}_{\mathbf{r}} = \mathcal{N} S$ are included. In particular, the mean field "quasi-particles" (bosons $\hat{\alpha}$) involve nonphysical density fluctuations of Schwinger bosons. Actual excitations are composites of an even number of quasi-particles. The local constraint is enforced through the dynamical fluctuations of the field λ and can be, at least in principle, reintroduced into the theory *via* a systematic $1/\mathcal{N}$ expansion [1].

8.A A single spin in an external field

We consider a single spin in an external field parallel to the z axis, with Hamiltonian $\hat{H} = -h\hat{S}^z$.

Free energy

The partition function is easily calculated using the basis $\{|m\rangle\}$ of eigenstates of $\hat{S}^z$, $\hat{S}^z|m\rangle = m|m\rangle$ $(-S \leq m \leq S)$,

$$Z = \mathrm{Tr}\, e^{\beta h \hat{S}^z} = \sum_{m=-S}^{S} e^{\beta h m} = \frac{\sinh\left[\beta h\left(S+\frac{1}{2}\right)\right]}{\sinh\left(\frac{\beta h}{2}\right)}. \tag{8.189}$$

The free energy is given by $F = -\frac{1}{\beta}\ln Z$ and the magnetization

$$m = -\frac{\partial F}{\partial h} = S B_S(\beta h S) \tag{8.190}$$

can be expressed in terms of the Brillouin function $B_S(x)$ [Eq. (8.27)].

Spin-spin correlation function

In imaginary time the connected propagator is defined by

$$G^{\nu\nu'}(\tau) = \langle T_\tau \hat{S}^\nu(\tau)\hat{S}^{\nu'}(0)\rangle - \langle \hat{S}^\nu\rangle\langle \hat{S}^{\nu'}\rangle, \tag{8.191}$$

where $\hat{S}^\nu(\tau) = e^{-\tau\hat{H}}\hat{S}^\nu e^{\tau\hat{H}}$ is the spin operator in the Heisenberg picture. $G^{\nu\nu'}$ is calculated by using the resolution of the identity $\sum_m |m\rangle\langle m| = 1$. For $\tau > 0$,

$$\begin{aligned} G^{zz}(\tau) &= \frac{1}{Z}\sum_m \langle m|e^{-(\beta-\tau)\hat{H}}\hat{S}^z|m\rangle\langle m|e^{\tau\hat{H}}\hat{S}^z|m\rangle - \langle \hat{S}^z\rangle^2 \\ &= \frac{1}{Z}\sum_m m^2 e^{\beta m h} - \left(\frac{1}{Z}\sum_m m\, e^{\beta m h}\right)^2, \end{aligned} \tag{8.192}$$

i.e.

$$G^{zz}(\tau) = -\frac{1}{\beta}\frac{\partial^2 F}{\partial h^2} = S^2 B_S'(\beta h S) \qquad (\tau > 0). \tag{8.193}$$

We deduce

$$G^{zz}(i\omega_n) = S^2 B_S'(\beta h S)\beta\delta_{\omega_n,0}. \tag{8.194}$$

A similar calculation gives[48]

$$\begin{aligned} G^{-+}(\tau) &= \frac{1}{Z}\sum_m e^{(\beta-\tau)mh+\tau(m+1)h}|\langle m+1|\hat{S}^+|m\rangle|^2 \\ &= \frac{e^{\tau h}}{e^{\beta h}-1} 2SB_S(\beta h S) \end{aligned} \tag{8.195}$$

[48] $\hat{S}^\pm = \hat{S}^x \pm i\hat{S}^y$ and $\hat{S}^\pm|m\rangle = [S(S+1) - m(m\pm 1)]^{1/2}|m\pm 1\rangle$.

for $\tau > 0$, i.e.

$$G^{-+}(i\omega_n) = \frac{2S}{i\omega_n + h} B_S(\beta h S), \tag{8.196}$$

and

$$G^{+-}(i\omega_n) = -\frac{2S}{i\omega_n - h} B_S(\beta h S). \tag{8.197}$$

From (8.196) and (8.197) we deduce

$$\begin{aligned} G^{xx}(i\omega_n) = G^{yy}(i\omega_n) &= \frac{h}{\omega_n^2 + h^2} S B_S(\beta h S), \\ G^{xy}(i\omega_n) = -G^{yx}(i\omega_n) &= -\frac{\omega_n}{\omega_n^2 + h^2} S B_S(\beta h S). \end{aligned} \tag{8.198}$$

8.B Spin-coherent states

It is useful to give a slightly different definition of the spin-coherent state $|\boldsymbol{\Omega}\rangle$ [Eq. (8.84)]. First we write

$$|\boldsymbol{\Omega}\rangle = e^{i\theta \mathbf{u}\cdot\hat{\mathbf{S}}}|S\rangle = e^{\frac{\theta}{2}e^{i\varphi}\hat{S}^- - \frac{\theta}{2}e^{-i\varphi}\hat{S}^+}|S\rangle, \tag{8.199}$$

where $\mathbf{u} = (\sin\varphi, -\cos\varphi, 0)$ and we have chosen the gauge $\psi = 0$. The SU(2) spin rotation operator in (8.199) corresponds to a rotation of angle $-\theta$ about the $\mathbf{u}$ axis. We then use[49]

$$e^{\frac{\theta}{2}e^{i\varphi}\hat{S}^- - \frac{\theta}{2}e^{-i\varphi}\hat{S}^+} = e^{z\hat{S}^-} e^{-\ln(1+|z|^2)\hat{S}^z} e^{-z^*\hat{S}^+} \quad \text{with} \quad z = \tan\left(\frac{\theta}{2}\right) e^{i\varphi} \tag{8.200}$$

and $e^{-z^*\hat{S}^+}|S\rangle = |S\rangle$ (which follows from $\hat{S}^+|S\rangle = 0$) to obtain

$$|\boldsymbol{\Omega}\rangle = e^{z\hat{S}^-} e^{-\ln(1+|z|^2)\hat{S}^z}|S\rangle = (1+|z|^2)^{-S} e^{z\hat{S}^-}|S\rangle. \tag{8.201}$$

Using

$$\hat{S}^-|S^z\rangle = [(S - S^z + 1)(S + S^z)]^{1/2}|S-1\rangle, \tag{8.202}$$

we finally obtain

$$|\boldsymbol{\Omega}\rangle = (1+|z|^2)^{-S} \sum_{p=0}^{2S} \sqrt{C_{2S}^p} z^p |S-p\rangle, \tag{8.203}$$

where $C_{2S}^p = (2S)!/[p!(2S-p)!]$.

It is then easy to show that the scalar product between two coherent states is given by

$$\begin{aligned} \langle \boldsymbol{\Omega}'|\boldsymbol{\Omega}\rangle &= \frac{(1+z'z)^{2S}}{(1+|z'|^2)^S(1+|z|^2)^S} \\ &= \left(\cos\frac{\theta'}{2}\cos\frac{\theta}{2} + e^{i(\varphi-\varphi')}\sin\frac{\theta'}{2}\sin\frac{\theta}{2}\right)^{2S} = \left(\frac{1}{2} + \frac{1}{2}\boldsymbol{\Omega}'\cdot\boldsymbol{\Omega}\right)^S \end{aligned} \tag{8.204}$$

[49] See Appendix A in Ref. [23] for a derivation of (8.200).

and the resolution of the identity reads

$$(2S+1)\int \frac{dz^* dz}{2i\pi} \frac{|z\rangle\langle z|}{(1+|z^2|)^2} = (2S+1)\int \frac{d\Omega}{4\pi} |\mathbf{\Omega}\rangle\langle\mathbf{\Omega}| = 1, \tag{8.205}$$

where $dz^* dz = 2id\Re(z)d\Im(z)$. From (8.203) we also deduce

$$\begin{aligned} \frac{\partial}{\partial\theta}|\mathbf{\Omega}\rangle &= (1+|z|^2)^{-S} \sum_{p=0}^{2S} \sqrt{C_{2S}^p} \left[-S|z| + \frac{p}{2|z|\left(\cos\frac{\theta}{2}\right)^2} \right] |S-p\rangle, \\ \frac{\partial}{\partial\varphi}|\mathbf{\Omega}\rangle &= (1+|z|^2)^{-S} \sum_{p=0}^{2S} \sqrt{C_{2S}^p}\, ipz^p |S-p\rangle, \end{aligned} \tag{8.206}$$

which leads to (8.96).

Guide to the bibliography

- For a general reference on quantum magnetism and the Heisenberg model, see Ref. [1]. The spin-$\frac{1}{2}$ Heisenberg model is reviewed in Ref. [2].
- Spinwaves in ferromagnets and antiferromagnets are discussed in a number of textbooks [1–3]. See also Ref. [24]
- Holstein-Primakoff bosons [4] are reviewed in Ref. [1].
- For a discussion of Berry phases [5], see [6–9].
- Spin-coherent states and the spin-coherent-state functional integral are discussed in [1, 10–12].
- The derivation of the nonlinear sigma model from the Heisenberg model [13–15] can be found in Refs. [1, 10]. The two-dimensional quantum nonlinear sigma model is discussed in section 12.3 and Ref. [16]. For a discussion of the role of the Berry phase term and the transition between the Néel state and the valence bond solid in two-dimensional Heisenberg antiferromagnets, see [10, 17].
- Quantum spin chains and the concept of Luttinger liquid are discussed in [10, 13, 14, 18, 19].
- The Schwinger-boson mean-field theory of the Heisenberg model is discussed in Refs. [1, 20, 21].

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