

Superuniversal transport near a (2 + 1)-dimensional quantum critical point

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(Received 16 May 2017; published 11 September 2017)

We compute the zero-temperature conductivity in the two-dimensional quantum $O(N)$ model using a nonperturbative functional renormalization-group approach. At the quantum critical point we find a universal conductivity σ^*/σ_Q (with $\sigma_Q = q^2/h$ the quantum of conductance and q the charge) in reasonable quantitative agreement with quantum Monte Carlo simulations and conformal bootstrap results. In the ordered phase the conductivity tensor is defined, when $N \geq 3$, by two independent elements, $\sigma_A(\omega)$ and $\sigma_B(\omega)$, respectively associated with $SO(N)$ rotations which do and do not change the direction of the order parameter. Whereas $\sigma_A(\omega \rightarrow 0)$ corresponds to the response of a superfluid (or perfect inductance), the numerical solution of the flow equations shows that $\lim_{\omega \rightarrow 0} \sigma_B(\omega)/\sigma_Q = \sigma_B^*/\sigma_Q$ is a superuniversal (i.e., N -independent) constant. These numerical results, as well as the known exact value $\sigma_B^*/\sigma_Q = \pi/8$ in the large- N limit, allow us to conjecture that $\sigma_B^*/\sigma_Q = \pi/8$ holds for all values of N , a result that can be understood as a consequence of gauge invariance and asymptotic freedom of the Goldstone bosons in the low-energy limit.

DOI: [10.1103/PhysRevB.96.100501](https://doi.org/10.1103/PhysRevB.96.100501)

Introduction. Understanding the physical properties of a system near a quantum phase transition constitutes an important problem in condensed-matter physics. This is particularly true of transport properties since strong fluctuations near the quantum critical point (QCP) often lead to the absence of well-defined quasiparticles and the breakdown of perturbation many-body theory [1].

In this Rapid Communication, we discuss the zero-temperature coherent transport near a relativistic (2 + 1)-dimensional QCP with an $O(N)$ -symmetric order parameter. We use a nonperturbative functional renormalization-group (NPRG) approach to compute the frequency-dependent conductivity in the quantum $O(N)$ model. The latter describes many condensed-matter systems with a relativistic effective low-energy dynamics: quantum antiferromagnets, bosons in optical lattices, Josephson junctions, etc. At the QCP we find a universal conductivity [2,3] σ^*/σ_Q (with $\sigma_Q = q^2/h$ the quantum of conductance and q the charge) in reasonable quantitative agreement with quantum Monte Carlo simulations [4–8] and conformal bootstrap results [9]. Our main result concerns the broken-symmetry phase, where the conductivity tensor has two independent elements σ_A and σ_B when $N \geq 3$, respectively associated with $SO(N)$ rotations which do and do not change the direction of the order parameter. Whereas $\sigma_A(\omega \rightarrow 0)$ corresponds to the response of a superfluid (or perfect inductance), the numerical solution of the NPRG equations together with the exact large- N result leads us to conjecture that $\sigma_B(\omega \rightarrow 0)/\sigma_Q$ takes the superuniversal (i.e., N -independent) value $\sigma_B^*/\sigma_Q = \pi/8$. We argue that this result is a consequence of gauge (rotation) invariance and asymptotic freedom in the infrared, i.e., the fact that Goldstone bosons become effectively noninteracting in the low-energy limit.

This conjecture has been anticipated in Ref. [10] using an approximate solution of the NPRG equations based on a derivative expansion of the scale-dependent effective action. However, because of infrared singularities which invalidate the derivative expansion at low energy, we could not obtain definite values for σ^*/σ_Q and σ_B^*/σ_Q . Here, we report results obtained from a different approximation of the NPRG equations which does not suffer from these limitations.

While universality is a generic consequence of the proximity of the QCP, universal quantities (e.g., critical exponents or scaling functions) in general depend on N . To our knowledge there are very few exceptions. The critical energy densities of $O(N)$ models on a d -dimensional lattice with long-range interactions are known to be all equal to the one of the Ising model [11]. The same is true for all $O(N)$ models on a one-dimensional lattice with nearest-neighbor interactions. It has been conjectured [12] that this superuniversality should hold for all d -dimensional $O(N)$ models but a firm numerical confirmation has not been provided so far [13].

Quantum $O(N)$ model and NPRG approach. The two-dimensional quantum $O(N)$ model is defined by the Euclidean action

$$S = \int_{\mathbf{x}} \left\{ \frac{1}{2} \sum_{\mu=0,x,y} (\partial_{\mu} \boldsymbol{\varphi})^2 + \frac{r_0}{2} \boldsymbol{\varphi}^2 + \frac{u_0}{4!N} (\boldsymbol{\varphi}^2)^2 \right\}, \quad (1)$$

where we use the notation $\mathbf{x} = (\mathbf{r}, \tau)$, $\int_{\mathbf{x}} = \int_0^{\beta} d\tau \int d^2r$, and $\partial_0 = \partial_{\tau}$. $\boldsymbol{\varphi}(\mathbf{x})$ is an N -component real field, \mathbf{r} a two-dimensional coordinate, $\tau \in [0, \beta]$ an imaginary time, and $\beta = 1/T$ the inverse temperature (we set $\hbar = k_B = 1$). r_0 and u_0 are temperature-independent coupling constants and the (bare) velocity of the $\boldsymbol{\varphi}$ field has been set to unity. The model is regularized by an ultraviolet cutoff Λ . Assuming u_0 fixed, there is a quantum phase transition between a disordered phase ($r_0 > r_{0c}$) and an ordered phase ($r_0 < r_{0c}$) where the $O(N)$ symmetry is spontaneously broken. The QCP at $r_0 = r_{0c}$ is in the universality class of the three-dimensional classical $O(N)$ model and the phase transition is governed by the three-dimensional Wilson-Fisher fixed point.

In the following we consider only the zero-temperature limit where the two-dimensional quantum model is equivalent to the three-dimensional classical model. We thus identify τ with a third spatial dimension so that $\mathbf{x} = (\mathbf{r}, \tau) \equiv (x, y, z)$. A correlation function $\chi(p_x, p_y, p_z)$ computed in the classical model then corresponds to the correlation function $\chi(p_x, p_y, i\omega_n)$ of the quantum model, with $\omega_n \equiv p_z$ a bosonic Matsubara frequency [14], and yields the retarded dynamical

correlation function $\chi^R(p_x, p_y, \omega)$ after analytical continuation $i\omega_n \rightarrow \omega + i0^+$.

The $O(N)$ symmetry of the action (1) implies the conservation of the total angular momentum and the existence of a conserved current. To compute the associated conductivity, we include in the model an external non-Abelian gauge field $A_\mu = A_\mu^a T^a$ (with an implicit sum over repeated discrete indices), where $\{T^a\}$ denotes a set of $SO(N)$ generators [made of $N(N-1)/2$ linearly independent skew-symmetric matrices]. This amounts to replacing the derivative ∂_μ in Eq. (1) by the covariant derivative $D_\mu = \partial_\mu - qA_\mu$ (we set the charge q equal to unity in the following and restore it, as well as \hbar , whenever necessary). This makes the action (1) invariant in the local gauge transformation $\phi' = O\phi$ and $A'_\mu = OA_\mu O^T + (\partial_\mu O)O^T$, where O is a space-dependent $SO(N)$ rotation. The current density $J_\mu^a(\mathbf{x}) = -\delta S/\delta A_\mu^a(\mathbf{x})$ is then expressed as [10]

$$J_\mu^a = j_\mu^a - A_\mu \phi \cdot T^a \phi, \quad j_\mu^a = \partial_\mu \phi \cdot T^a \phi, \quad (2)$$

where j_μ^a denotes the ‘‘paramagnetic’’ part. For $N = 2$, there is a single generator T , which can be chosen as minus the antisymmetric tensor ϵ_{ij} [10], and we recover the standard expression $j_\mu = -i[\psi^* \partial_\mu \psi - (\partial_\mu \psi^*)\psi]$ of the current density of bosons described by a complex field $\psi = (\varphi_1 + i\varphi_2)/\sqrt{2}$. For $N = 3$, there are three generators $-iS^1$, $-iS^2$, and $-iS^3$ related to spin-one matrices S^i . One then finds $j_\mu^i = -\epsilon_{ijk}(\partial_\mu \varphi_j)\varphi_k$ (ϵ_{ijk} is the antisymmetric tensor), in agreement with the continuum limit of spin currents defined in lattice models [15].

The frequency-dependent conductivity of the quantum model is defined as the linear response to the gauge field, i.e.,

$$\sigma_{\mu\nu}^{ab}(\omega) = -\frac{i}{(\omega + i0^+)} K_{\mu\nu}^{abR}(\omega). \quad (3)$$

K^R is the retarded part of the correlation function

$$K_{\mu\nu}^{ab}(i\omega_n) = \Pi_{\mu\nu}^{ab}(i\omega_n) - \delta_{\mu\nu} \langle T^a \phi \cdot T^b \phi \rangle, \quad (4)$$

where we have set the momentum to zero and $\Pi_{\mu\nu}^{ab} = \langle j_\mu^a j_\nu^b \rangle$ is the paramagnetic current-current correlation function. The conductivity having a vanishing scaling dimension in two space dimensions, it satisfies [2,16]

$$\sigma_{\mu\nu}^{ab}(\omega) = \sigma_Q \Sigma_{\mu\nu\pm}^{ab} \left(\frac{\omega + i0^+}{\Delta} \right), \quad (5)$$

where $\Sigma_{\mu\nu\pm}^{ab}$ is a universal scaling function (the index $+/-$ refers to the disordered/ordered phase), and Δ a characteristic zero-temperature energy scale which measures the distance to the QCP. In the disordered phase, we take Δ to be equal to the excitation gap. In the ordered phase, we choose Δ to be given by the excitation gap in the disordered phase at the point located symmetrically with respect to the QCP (i.e., corresponding to the same value of $|r_0 - r_{0c}|$). The conductivity tensor is diagonal in the disordered phase so that a single scaling function Σ_+ has to be considered. In the ordered phase, it has only two independent elements, σ_A and σ_B , respectively associated with $SO(N)$ rotations which do and do not change the direction of the order parameter [10].

(For $N = 2$ there is only one generator and the conductivity is diagonal also in the ordered phase.)

The strategy of the NPRG approach is to build a family of models indexed by a momentum scale k such that fluctuations are smoothly taken into account as k is lowered from the microscopic scale Λ down to 0 [17–19]. This is achieved by adding to the action $S[\phi, \mathbf{A}]$ the gauge-invariant infrared regulator term [10]

$$\Delta S_k[\phi, \mathbf{A}] = \frac{1}{2} \int_{\mathbf{x}} \phi \cdot R_k(-D^2)\phi, \quad (6)$$

where $D^2 = D_\mu D_\mu$. The partition function

$$\mathcal{Z}_k[\mathbf{J}, \mathbf{A}] = \int \mathcal{D}[\phi] e^{-S[\phi, \mathbf{A}] - \Delta S_k[\phi, \mathbf{A}] + \int_{\mathbf{x}} \mathbf{J} \cdot \phi} \quad (7)$$

is now k dependent. Here, \mathbf{J} is an external source which couples linearly to the ϕ field. The order parameter $\phi_k[\mathbf{x}; \mathbf{J}, \mathbf{A}] = \delta \ln \mathcal{Z}_k[\mathbf{J}, \mathbf{A}]/\delta \mathbf{J}(\mathbf{x}) = \langle \phi(\mathbf{x}) \rangle$ is a functional of both \mathbf{J} and \mathbf{A} . The scale-dependent effective action

$$\Gamma_k[\phi, \mathbf{A}] = -\ln \mathcal{Z}_k[\mathbf{J}, \mathbf{A}] + \int_{\mathbf{x}} \mathbf{J} \cdot \phi - \Delta S_k[\phi, \mathbf{A}] \quad (8)$$

is defined as a (slightly modified) Legendre transform of $-\ln \mathcal{Z}_k[\mathbf{J}, \mathbf{A}]$, where the linear source $\mathbf{J} \equiv \mathbf{J}_k[\phi, \mathbf{A}]$ is now considered as a functional of ϕ and \mathbf{A} . Assuming that fluctuations are completely frozen by the ΔS_k term when $k = \Lambda$, $\Gamma_\Lambda[\phi, \mathbf{A}] = S[\phi, \mathbf{A}]$. On the other hand, the effective action of the original model, defined by the action $S[\phi, \mathbf{A}]$, is given by $\Gamma_{k=0}$ provided that $R_{k=0}$ vanishes. The variation of the effective action with k is given by Wetterich’s equation [20]

$$\partial_k \Gamma_k[\phi, \mathbf{A}] = \frac{1}{2} \text{Tr} \{ \partial_k \mathcal{R}_k[\mathbf{A}] (\Gamma_k^{(2,0)}[\phi, \mathbf{A}] + \mathcal{R}_k[\mathbf{A}])^{-1} \}, \quad (9)$$

where $\Gamma_k^{(2,0)}[\phi, \mathbf{A}]$ and $\mathcal{R}_k[\mathbf{A}]$ denote the second-order functional derivative with respect to ϕ of $\Gamma_k[\phi, \mathbf{A}]$ and $\Delta S_k[\phi, \mathbf{A}]$, respectively. In Fourier space, the trace involves a sum over momenta as well as the $O(N)$ index of the ϕ field. The conductivity of the quantum model is calculated using

$$K_{\mu\nu}^{ab}(\mathbf{p}, i\omega_n) = -\Gamma_{k=0, \mu\nu}^{(0,2)ab}(\mathbf{p}, i\omega_n), \quad (10)$$

where $\Gamma_k^{(0,2)}$ is the second-order functional derivative of $\Gamma_k[\phi, \mathbf{A}]$ with respect to \mathbf{A} , evaluated for $\mathbf{A} = \mathbf{0}$ and in the uniform time-independent field configuration ϕ which minimizes the effective action $\Gamma_{k=0}[\phi, \mathbf{A} = \mathbf{0}]$ [10].

To solve Eq. (9) we consider the following gauge-invariant ansatz,

$$\begin{aligned} \Gamma_k[\phi, \mathbf{A}] = \int_{\mathbf{x}} \{ & U_k(\rho) + \frac{1}{2} D_\mu \phi \cdot Z_k(-D^2) D_\mu \phi \\ & + \frac{1}{4} (\partial_\mu \rho) Y_k(-\partial^2) (\partial_\mu \rho) + \frac{1}{4} F_{\mu\nu}^a X_{1,k}(-D^2) F_{\mu\nu}^a \\ & + \frac{1}{4} F_{\mu\nu}^a T^a \phi \cdot X_{2,k}(-D^2) F_{\mu\nu}^b T^b \phi \}, \end{aligned} \quad (11)$$

which, in addition to the effective potential $U_k(\rho = \phi^2/2)$, involves four functions of momentum: $Z_k(\mathbf{q}^2)$, $Y_k(\mathbf{q}^2)$, $X_{1,k}(\mathbf{q}^2)$, and $X_{2,k}(\mathbf{q}^2)$. This approximation, which we dub LPA'', has been used in the past to compute the critical indices and the momentum dependence of correlation functions in the $O(N)$ model in the absence of the gauge field [21,22]. For

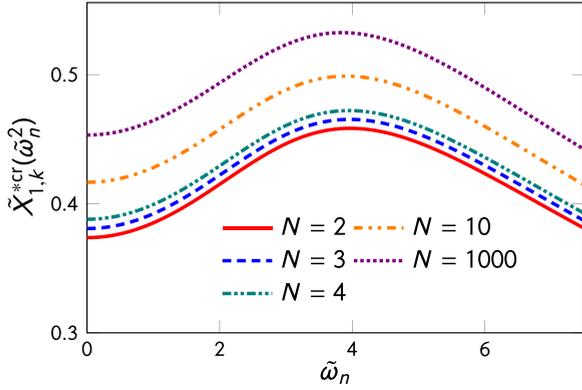


FIG. 1. Fixed-point function $\tilde{X}_{1,k}^{*cr}(\tilde{\omega}_n^2)$ at the QCP for various values of N .

$Z_k(\mathbf{q}^2) \equiv Z_k$ and $Y_k(\mathbf{q}^2) = 0$ it reduces to the LPA', an improvement of the local potential approximation (LPA) which includes a field-renormalization factor Z_k [17,18]. We denote by $\rho_{0,k}$ the value of ρ at the minimum of the effective potential. Spontaneous breaking of the $O(N)$ symmetry is characterized by a nonvanishing value of $\rho_{0,k}$ for $k \rightarrow 0$.

From Eqs. (11) and (9) we obtain RG equations for the functions U_k , Z_k , Y_k , $X_{1,k}$, and $X_{2,k}$, and in turn for the vertex

$$\Gamma_{k,\mu\nu}^{(0,2)ab}(\mathbf{p} = 0, i\omega_n) = \delta_{\mu\nu} \delta_{ab} \omega_n^2 X_{1,k}(\omega_n^2) + \delta_{\mu\nu} (T^a \boldsymbol{\phi}) \cdot (T^b \boldsymbol{\phi}) \times [Z_k(\omega_n^2) + \omega_n^2 X_{2,k}(\omega_n^2)], \quad (12)$$

which determines the conductivity [Eqs. (3) and (10)]. Here, $\boldsymbol{\phi}$ denotes the order parameter with modulus $|\boldsymbol{\phi}| = \sqrt{2\rho_{0,k}}$ and arbitrary direction. For the numerical solution of the RG equations we consider dimensionless variables expressing all quantities in units of the running momentum scale k so that the QCP manifests itself as a fixed point of the RG equations. The latter are solved numerically with the explicit Euler method and a discretization of the (properly adimensionalized) ρ and ω_n variables, and an exponential regulator function $R_k(\mathbf{q}^2) = \alpha Z_k(0) \mathbf{q}^2 / (e^{\mathbf{q}^2/k^2} - 1)$ with an adjustable parameter α as in Ref. [10].

Conductivity. At the QCP we expect $\sigma(\omega \rightarrow 0)/\sigma_Q$ to take a nonzero universal value σ^*/σ_Q [2,3]. The k -dependent conductivity $\sigma_k(i\omega_n)$, as a function of the Matsubara frequency ω_n , is given by

$$\sigma_k(i\omega_n) = 2\pi\sigma_Q\omega_n X_{1,k}(\omega_n^2) = 2\pi\sigma_Q\tilde{\omega}_n \tilde{X}_{1,k}(\tilde{\omega}_n^2) \quad (13)$$

when the order parameter vanishes ($\rho_{0,k} = 0$). Here, $\tilde{\omega}_n = \omega_n/k$ is a dimensionless frequency and $\tilde{X}_{1,k}(\tilde{\omega}_n^2) = kX_{1,k}(\omega_n^2)$ a dimensionless function of $\tilde{\omega}_n^2$. At the QCP, the function $\tilde{X}_{1,k}$ reaches a k -independent fixed-point value $\tilde{X}_{1,k}^{*cr}$ and the conductivity takes the form $\sigma_k(i\omega_n) = 2\pi\sigma_Q\tilde{\omega}_n \tilde{X}_{1,k}^{*cr}(\tilde{\omega}_n^2)$. The low-frequency universal conductivity is obtained by taking first the limit $k \rightarrow 0$ and then $\omega_n \rightarrow 0$, i.e., $\tilde{\omega}_n \rightarrow \infty$: $\sigma^*/2\pi\sigma_Q = \lim_{\tilde{\omega}_n \rightarrow \infty} \tilde{\omega}_n \tilde{X}_{1,k}^{*cr}(\tilde{\omega}_n^2)$ is thus determined by the $1/\tilde{\omega}_n$ behavior of $\tilde{X}_{1,k}^{*cr}(\tilde{\omega}_n^2)$ at high frequencies (Fig. 1). Note that this $1/\tilde{\omega}_n$ high-frequency tail, which corresponds to a $1/\omega_n$ divergence of $X_{1,k=0}(\omega_n^2)$ for $\omega_n \rightarrow 0$, is responsible for the breakdown of the derivative expansion of Γ_k used in Ref. [10]. The value of σ^* depends weakly on the regulator, through the arbitrary parameter α [23]. This dependence on α decreases as N increases, and

TABLE I. Universal conductivity σ^*/σ_Q at the QCP [3], obtained with a regulator parameter value of $\alpha = 2.25$, compared to results obtained from quantum Monte Carlo simulations [4–8] (QMC) and conformal bootstrap [9] (CB). The exact value for $N \rightarrow \infty$ is $\pi/8 \simeq 0.3927$.

N	NPRG	QMC	CB
2	0.3218	0.355–0.361	0.3554(6)
3	0.3285		
4	0.3350		
10	0.3599		
1000	0.3927		

at $N = \infty$ the results do not depend on α and the exact value $\sigma^*/\sigma_Q = \pi/8$ is recovered [1]. The universal conductivity σ^* is shown in Table I for various values of N . For $N = 2$ we find a value in reasonable agreement with (although 10% smaller than) results from QMC [4–8] and conformal bootstrap [9].

In the disordered phase, away from the QCP, Eq. (13) still holds since the order parameter vanishes. In Fig. 2 we show the real-frequency conductivity $\sigma(\omega)$ obtained from $\sigma_{k=0}(i\omega_n)$ by analytical continuation using Padé approximants [24], a method which has proven to be reliable in the NPRG approach [10,25,26]. As expected, the system is insulating. The real part of the conductivity vanishes below the two-particle excitation gap 2Δ and the system behaves as a perfect capacitor for $|\omega| \ll \Delta$, i.e., $\sigma(\omega) \simeq -iC_{\text{dis}}\omega$, with capacitance (per unit area) $C_{\text{dis}} = 2\pi\hbar\sigma_Q X_{1,k=0}(\omega_n^2 = 0)$. Note that for large N , there is a discrepancy between the exact solution and our computation. Indeed, unlike at the QCP and in the ordered

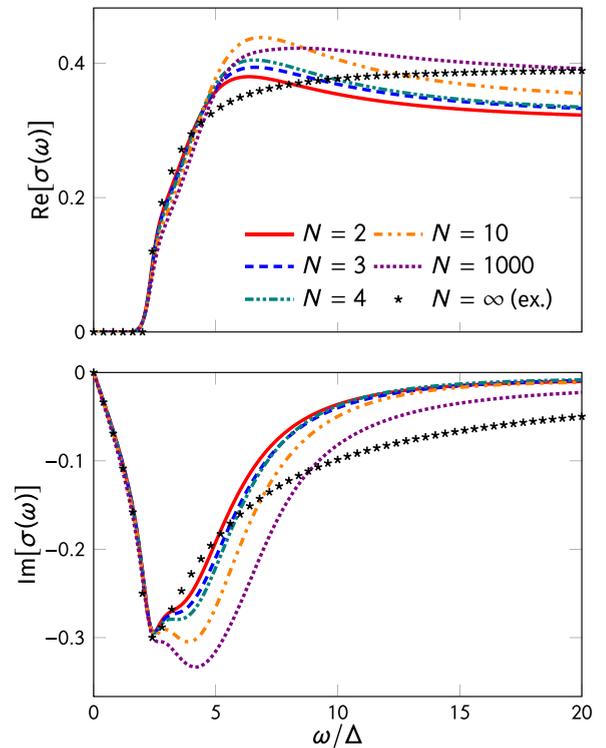


FIG. 2. Real and imaginary parts of the conductivity $\sigma(\omega)$ in the disordered phase for various values of N . Black stars show the exact large- N solution.

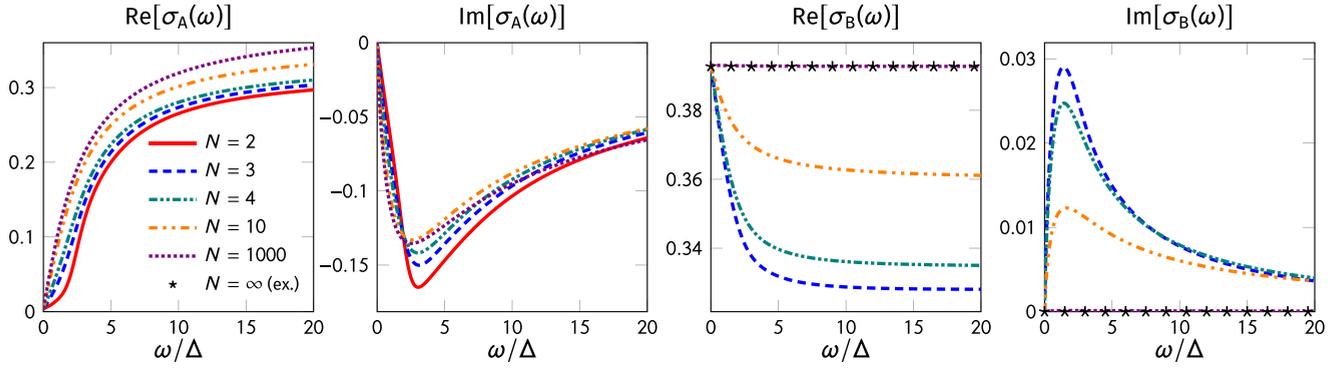


FIG. 3. Real and imaginary parts of the conductivity in the ordered phase for various values of N . Left: $\sigma_A(\omega)$ with the superfluid contribution subtracted. Right: $\sigma_B(\omega)$. Black stars show the exact large- N solution $\sigma_B(\omega) = \pi/8$.

phase, in the disordered phase the LPA'' does not reproduce the large- N solution [27]. Furthermore, the analytic continuation is made difficult by the singularity at $\omega = 2\Delta$ so that the frequency dependence of $\sigma(\omega)$ above 2Δ should be taken with caution.

Let us finally discuss the two elements, σ_A and σ_B , of the conductivity tensor in the ordered phase where the $O(N)$ symmetry is spontaneously broken:

$$\begin{aligned}\sigma_{A,k}(i\omega_n) &= 2\pi\sigma_Q \left\{ \omega_n X_{1,k}(\omega_n^2) + (2\rho_{0,k}/\hbar) \right. \\ &\quad \left. \times [Z_k(\omega_n^2)/\omega_n + \omega_n X_{2,k}(\omega_n^2)] \right\}, \\ \sigma_{B,k}(i\omega_n) &= 2\pi\sigma_Q \omega_n X_{1,k}(\omega_n^2).\end{aligned}\quad (14)$$

At low frequencies, $\sigma_{A,k}(i\omega_n)/2\pi\sigma_Q \simeq 2\rho_{0,k}Z_k(0)/\hbar\omega_n$ is characteristic of a superfluid system with stiffness $\rho_{s,k} = 2\rho_{0,k}Z_k(0)$ (i.e., a perfect inductor with inductance $L_{\text{ord}} = \hbar/2\pi\sigma_Q\rho_s$). $\sigma_A(\omega)$, with the superfluid contribution subtracted, is shown in Fig. 3. Our results seem to indicate the absence of a constant $\mathcal{O}(\omega_n^0)$ term in agreement with the predictions of perturbation theory [28]. Furthermore, we see a marked difference in the low-frequency behavior of the real part of the conductivity between the cases $N = 2$ and $N \neq 2$, but our numerical results are not precise enough to resolve the low-frequency power laws (predicted [28] to be ω and

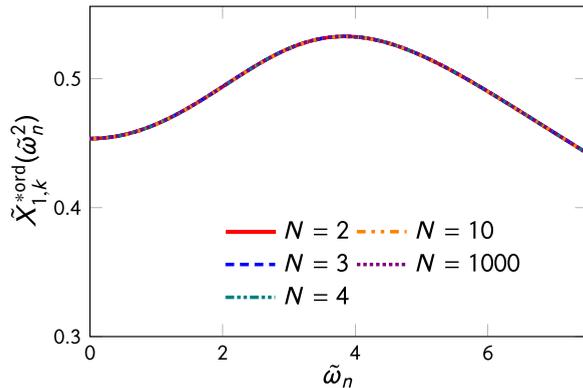


FIG. 4. Fixed-point function $\tilde{X}_{1,k}^{*\text{ord}}(\tilde{\omega}_n^2)$ in the ordered phase for various values of N . The universal conductivity σ_B^* is determined by the $1/\tilde{\omega}_n$ high-frequency tail and the collapse of the curves indicates that σ_B^* is N independent.

ω^5 for $N \neq 2$ and $N = 2$, respectively). On the other hand, we find that $\sigma_B(\omega)$ reaches a nonzero universal value σ_B^* in the limit $\omega \rightarrow 0$ (Fig. 3). As for the conductivity at the QCP, this universal value is determined by the $1/\tilde{\omega}_n$ high-frequency tail of the fixed-point value $\tilde{X}_{1,k}^{*\text{ord}}(\tilde{\omega}_n^2)$ of the dimensionless function $\tilde{X}_{1,k}(\tilde{\omega}_n^2)$. Quite surprisingly, and contrary to $\tilde{X}_{1,k}^{*\text{cr}}$ (Fig. 1), $\tilde{X}_{1,k}^{*\text{ord}}$ turns out to be N independent: The relative change in $\tilde{X}_{1,k}^{*\text{ord}}$ is less than 10^{-6} when N varies (Fig. 4). Noting that the obtained value $\sigma_B^*/\sigma_Q \simeq 0.3927$ is equal to the large- N result [10,29] $\pi/8$ within numerical precision, we conjecture that $\sigma_B^*/\sigma_Q = \pi/8$ for all values of N .

This result can be simply understood by noting that Goldstone bosons become effectively noninteracting in the infrared limit [30]. Using the renormalized Goldstone-boson propagator $G_T(\mathbf{p}, i\omega_n) = [Z(\mathbf{p}^2 + \omega_n^2)]^{-1}$, where $Z = \rho_s/2\rho_0$ is the field renormalization factor, and noting Z_B the renormalization factor associated with the boson-gauge-field interaction for a class B generator, an elementary calculation then gives $\sigma_B^*/\sigma_Q = (Z_B/Z)^2\pi/8$. Gauge invariance implies that Z_B and Z are not independent but related by the Ward identity $Z_B = Z$, so that we finally obtain $\sigma_B^*/\sigma_Q = \pi/8$, in agreement with the NPRG result [31].

Conclusion. We have determined the frequency-dependent zero-temperature conductivity near a relativistic $(2+1)$ -dimensional QCP with an $O(N)$ -symmetric order parameter. Our results are obtained using the LPA'', an approximation of the exact RG flow equation satisfied by the effective action which respects the local gauge invariance of the theory while retaining the full momentum/frequency dependence of the vertices. Besides the frequency dependence of the conductivity both in the ordered and disordered phases, our main result is the conjecture that $\sigma_B(\omega \rightarrow 0)/\sigma_Q$ takes the superuniversal (N -independent) value $\sigma_B^*/\sigma_Q = \pi/8$ [3]. This result could in principle be confirmed experimentally in two-dimensional quantum antiferromagnets, where both quantum criticality and the Higgs amplitude mode have been recently observed [32,33], although the frequency-dependent spin conductivity has not been measured so far. A natural continuation of this work would be to extend the NPRG procedure to finite temperatures to investigate both the collisionless ($\omega \gg T$) and hydrodynamic ($\omega \ll T$) regimes, the latter being inaccessible at zero temperature.

Acknowledgment. We thank N. Defenu for pointing out Refs. [12,13] and P. Kopietz for correspondence.

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