Critical properties of Anderson localization in high-dimension

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Outline of the talk

- **Introduction**
  - Anderson model
  - Localization transition
  - Known results & open problems
- **Numerics in $d = 3, \ldots, 6$**
  - Exact diagonalization & transfer matrix
  - Finite size corrections
- **Strong Disorder RG approach**
- **Summary of the results**
  - Critical exponent
  - Critical values
  - Multifractality
- **Conclusions & perspectives**

Anderson model

Non-interacting (spinless) electrons in a disordered potential

$$\mathcal{H} = -\sum_i \epsilon_i c_i^\dagger c_i - t \sum_{\langle i,j \rangle} \left(c_i^\dagger c_j + c_j^\dagger c_i \right)$$ Anderson '58

$\langle i, j \rangle$ nearest neighbors on a d-dimensional hypercube

ϵ_i iid uniformly distributed in $[-W/2, W/2]$

Random Matrix model:

$$\mathcal{H}_{ij} = -\epsilon_i \delta_{ij} - t\, \mathcal{C}_{ij}^{(d)}$$

Diagonal disorder + Connectivity matrix of the d-dimensional hypercube

Localization transition for strong enough disorder

Recent experiments in $1d$ (atomic gases) and $3d$ (sound waves)

Aspect & Inguscio '09; Greiner & al '02; Hu & al '08

Localization transition in low dimensions

Good understanding of AL in low dimensional systems

- All states are localized in $d = 1$ and $d = 2$
- Scaling theory of localization Abrahams & al ’79
- $d = 2$ is the lower critical dimension (“weak localization”)
- Field theory description of the replicated NLσM in $2 + \epsilon$ (transition at weak disorder) Wegner ’79; Brézin & al ’80; Hikami ‘92
- Multifractal spectra at AL using Functional RG
  Foster & al ’09;
  Carpentier & Le Doussal ‘00

No analytical result in higher dimensions

Numerical methods are still at the core of the advances in this topic

Localization transition in $d = 3$

$W_c \simeq 16.35$

Metallic phase | **Insulating phase**

- Extended wave-functions
  $|\psi_i^{(\alpha)}|^2 \sim 1/L^d$ on all sites
  $$\Upsilon_\alpha = \sum_i |\psi_i^{(\alpha)}|^4 \sim \mathrm{cst}/L^d$$
- Energy transport $(\sigma > 0)$
- Level repulsion
  GOE statistics

- Localized wave-functions
  $|\psi_i^{(\alpha)}|^2 \sim O(1)$ on $O(1)$ sites
  $$\Upsilon_\alpha = \sum_i |\psi_i^{(\alpha)}|^4 \sim O(1)$$
- No energy transport $(\sigma = 0)$
- No level repulsion
  Poisson statistics

$$\xi \sim |W - W_c|^{-\nu} \qquad \nu \simeq 1.57$$

Slevin & al '09

Non-universal level
statistics and multifactality
at the critical point

Open problems & motivations

Much less is known in higher dimensions

- **Upper critical dimension?**

- **Understanding the $d = \infty$ limit!**

  - Anderson model on the Bethe lattice (and Lévy matrices) show **huge finite size effects** and the presence of a **critical crossover region**
  - **Intermediate non-ergodic delocalized phase?**

  Biroli & al '12; De Luca & al '14; Tarquini & al '15;
  Tikhonov & al '16; Altshuler & al '16

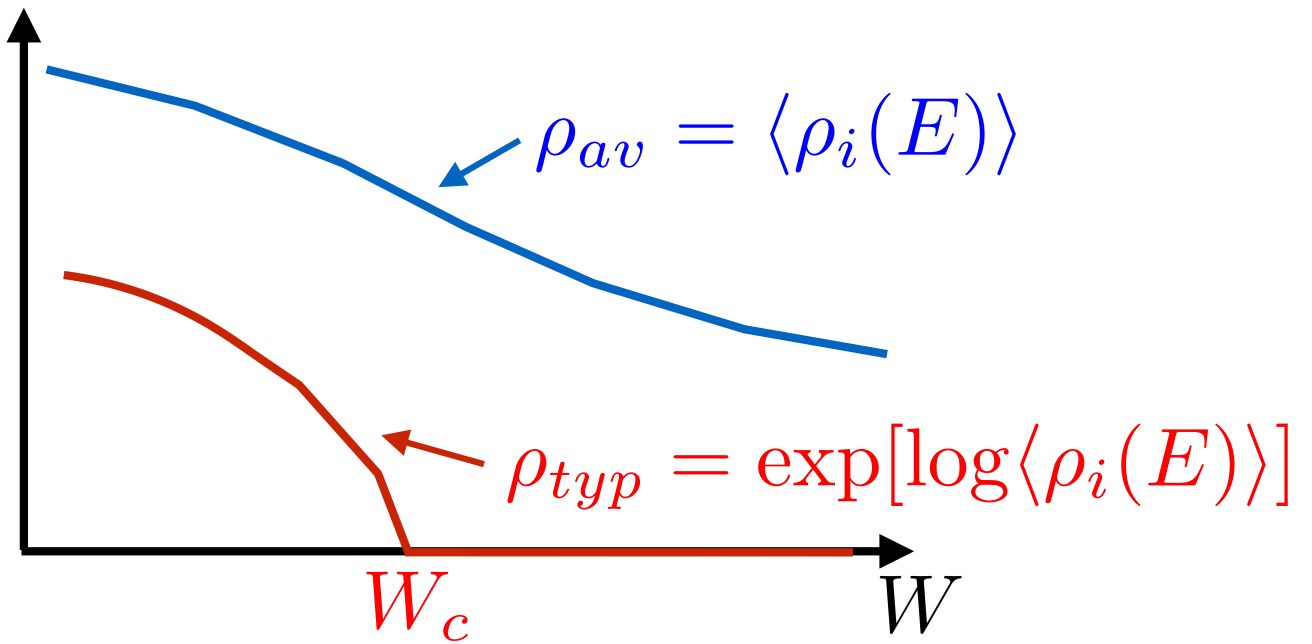
- **Relation with MBL** (AL in the Fock space)

Basko & al '06; Altshuler & al '97

No (conventional) symmetry breaking

Order parameter??

Local DOS: $\rho_i(E) = \frac{1}{\pi} \lim_{\eta \to 0} \mathrm{Im} G_{ii} = \sum_\alpha \delta(E - E_\alpha) |\psi_i^{(\alpha)}|^2$



"Ortodox phase transition theory will be of little help to us for the time being, and thus the great body of literature in the field is simply irrelevant"

Anderson '79

Numerics in $d = 3, \ldots, 6$

- **Exact diagonalization**

  Level statistics, IPR,
  statistics of wave-functions amplitudes, …

  Computation time $\sim L^{3d}$

  $L_{\max} = 30$ in $d = 3$; $L_{\max} = 13$ in $d = 4$;

  $L_{\max} = 8$ in $d = 5$; $L_{\max} = 5$ in $d = 6$

- **Transfer matrix approach**

  Lyapunov exponent, conductivity, …

  Computation time $\sim L^{3d-2}$

  $L_{\max} = 64$ in $d = 3$; $L_{\max} = 16$ in $d = 4$;

  $L_{\max} = 9$ in $d = 5$; $L_{\max} = 6$ in $d = 6$

Level statistics

- Distribution of the ratio of adjacent gaps
  Oganesyan & Huse '07

$$s_\alpha = E_{\alpha+1} - E_\alpha > 0 \qquad 0 \leq r_\alpha = \frac{\min\{s_\alpha, s_{\alpha-1}\}}{\max\{s_\alpha, s_{\alpha-1}\}} \leq 1$$

Universal distributions for GOE and Poisson statistics

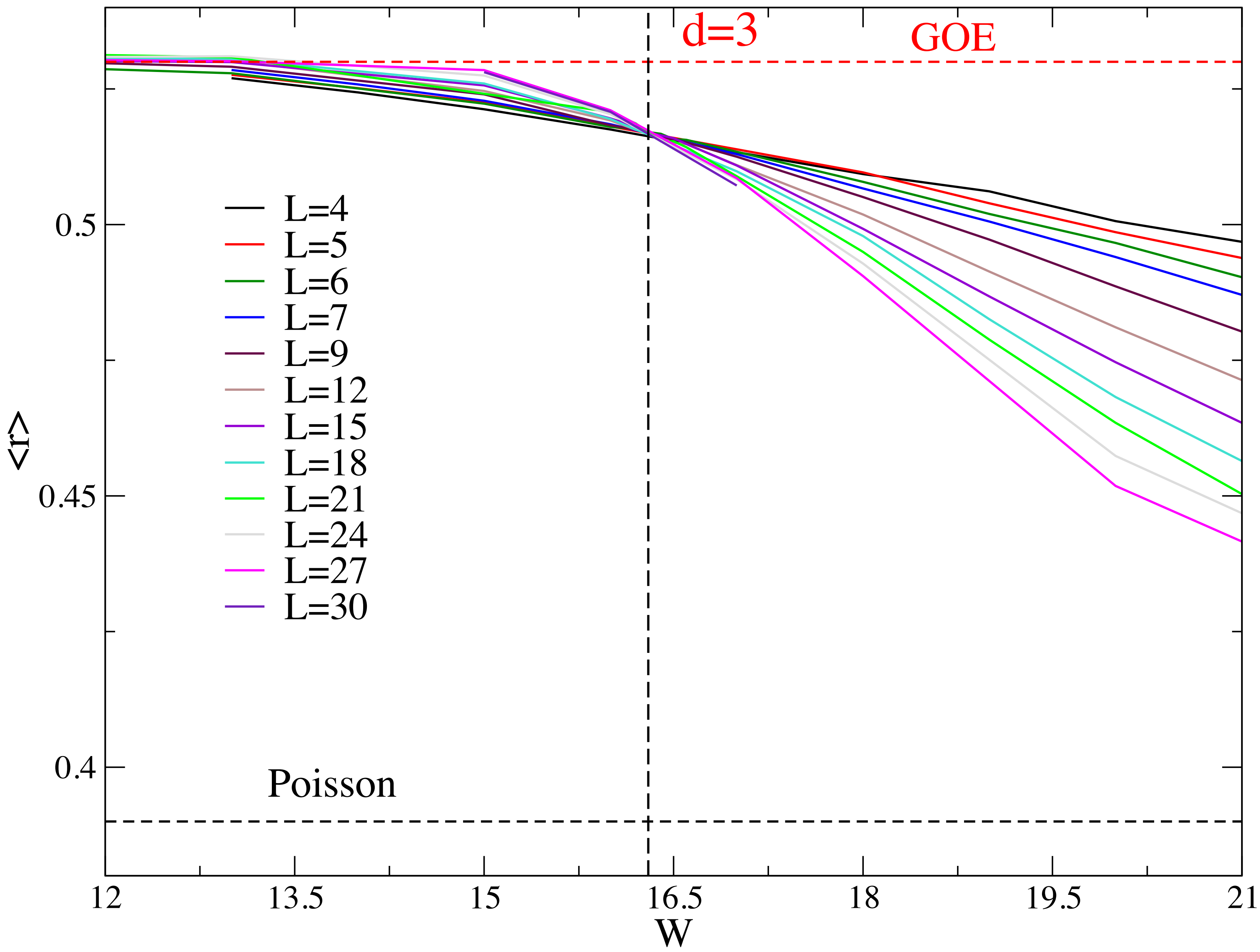
$$\Pi_{GOE}(r) \longrightarrow \langle r \rangle_{GOE} = 0.53$$

$$\Pi_P(r) = \frac{2}{(1+r)^2} \longrightarrow \langle r \rangle_P = 0.39$$

- Overlap between subsequent wave-functions

$$q_\alpha = \sum_i |\psi_i^{(\alpha)}||\psi_i^{(\alpha+1)}| \qquad \langle q \rangle_{GOE} = 2/\pi \qquad \langle q \rangle_P \to 0$$

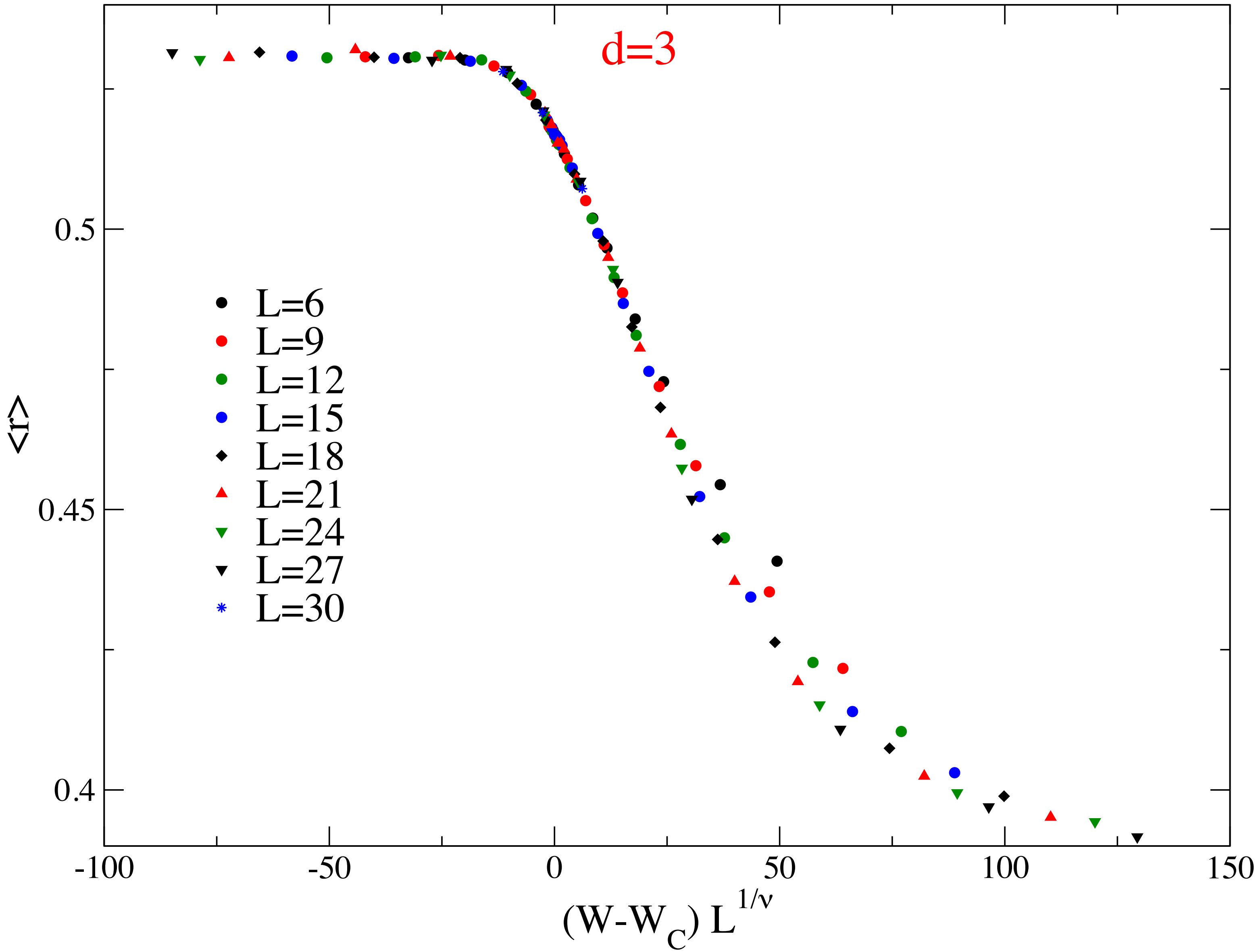
Level statistics in $d = 3$



Finite Size Scaling

$$\langle r \rangle = f(|W - W_c| L^{\frac{1}{\nu}})$$

MacKinnon & Kramer '80



$\nu \simeq 1.57$

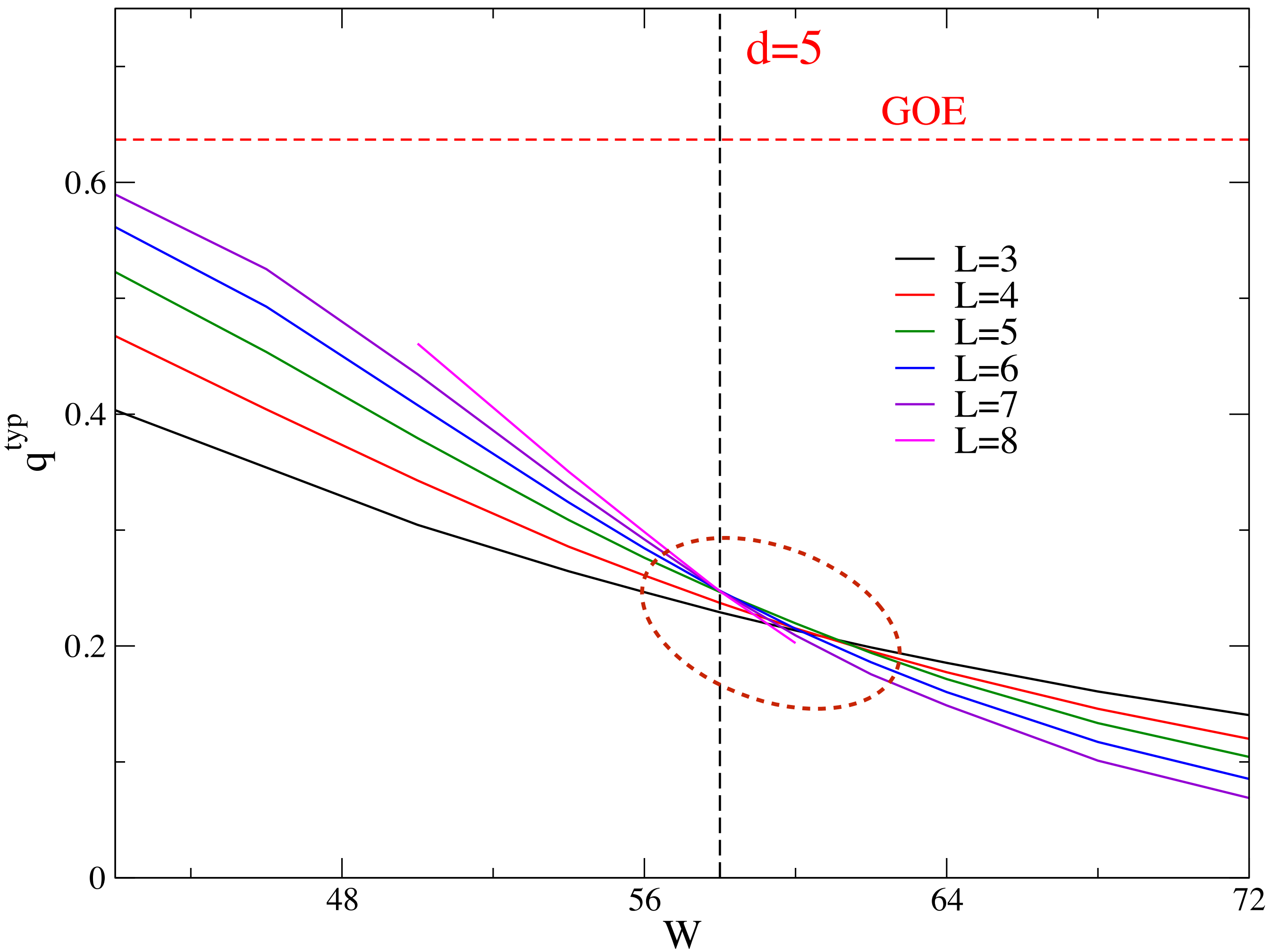
$W_c \simeq 16.35$

Slevin & al '09

Finite Size Corrections $(d = 5)$

Finite size effects becomes important in higher dimension

$$q^{typ} = F(|W - W_c|L^{\frac{1}{\nu}}, \phi L^a) \qquad (a < 0)$$
$$\simeq f_0(|W - W_c|L^{\frac{1}{\nu}}) + \phi L^a f_1(|W - W_c|L^{\frac{1}{\nu}})$$



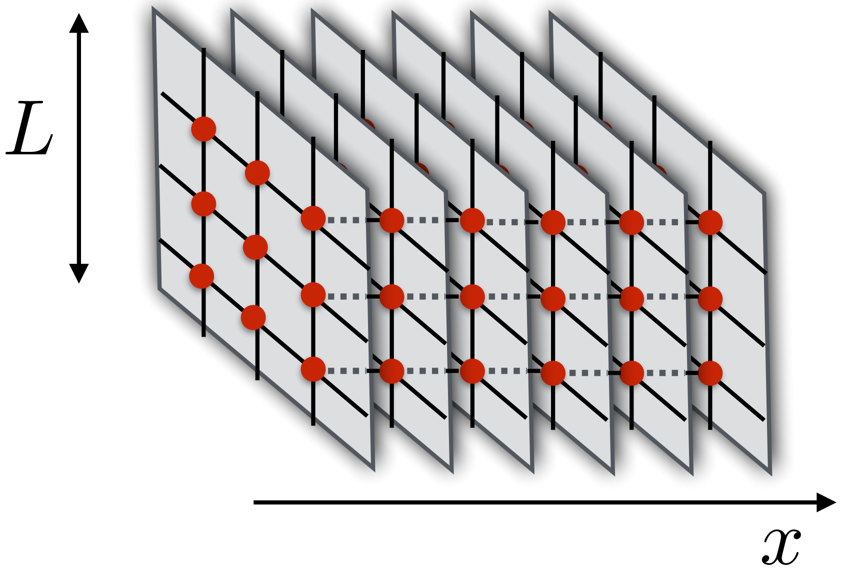
$\nu \simeq 0.95$

$W_c \simeq 57.5$

$a \simeq -1$

Ueoka & Slevin '14

Transfer Matrix



Quasi-$1d$ dimensional bar of cross-section L^{d-1}

$$[G^{(x+1)}]^{-1}_{ij} = \epsilon_{x,i}\delta_{ij} + t\,\mathcal{C}^{(d-1)}_{ij} - t^2 G^{(x)}_{ij}$$

Initial conditions: $[G^{(x=0)}]^{-1}_{ij} = (\epsilon_{0,i} + i\eta)\delta_{ij} + t\,\mathcal{C}^{(d-1)}_{ij}$

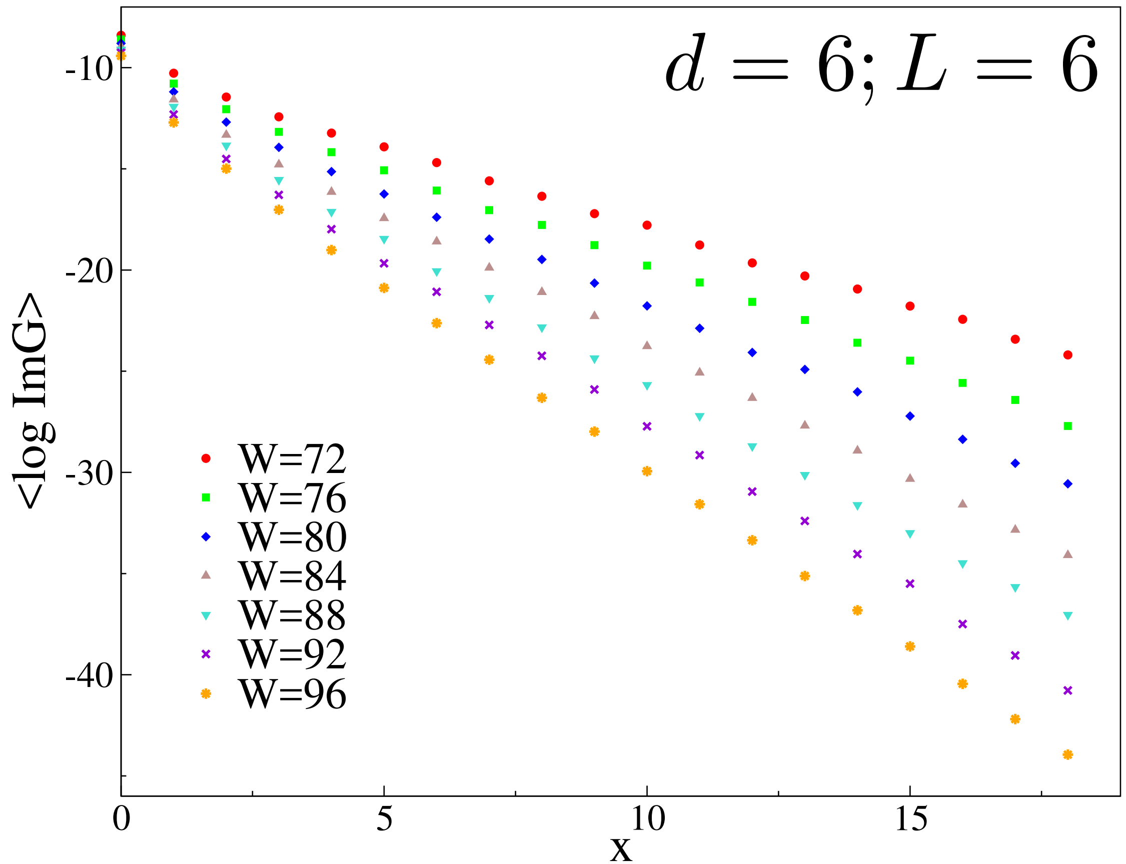
Inversion of $G^{(x)}$ by LU decomposition $[\sim L^{3(d-1)}]$

Propagation of $\mathrm{Im}G$ yields the quasi-$1d$ localization length ξ_{1d} (Lyapunov exponent)

Quasi-$1d$ localization length

ξ_{1d} saturates to ξ in the localized regime

ξ_{1d} diverges as σL^{d-1} in the metallic phase

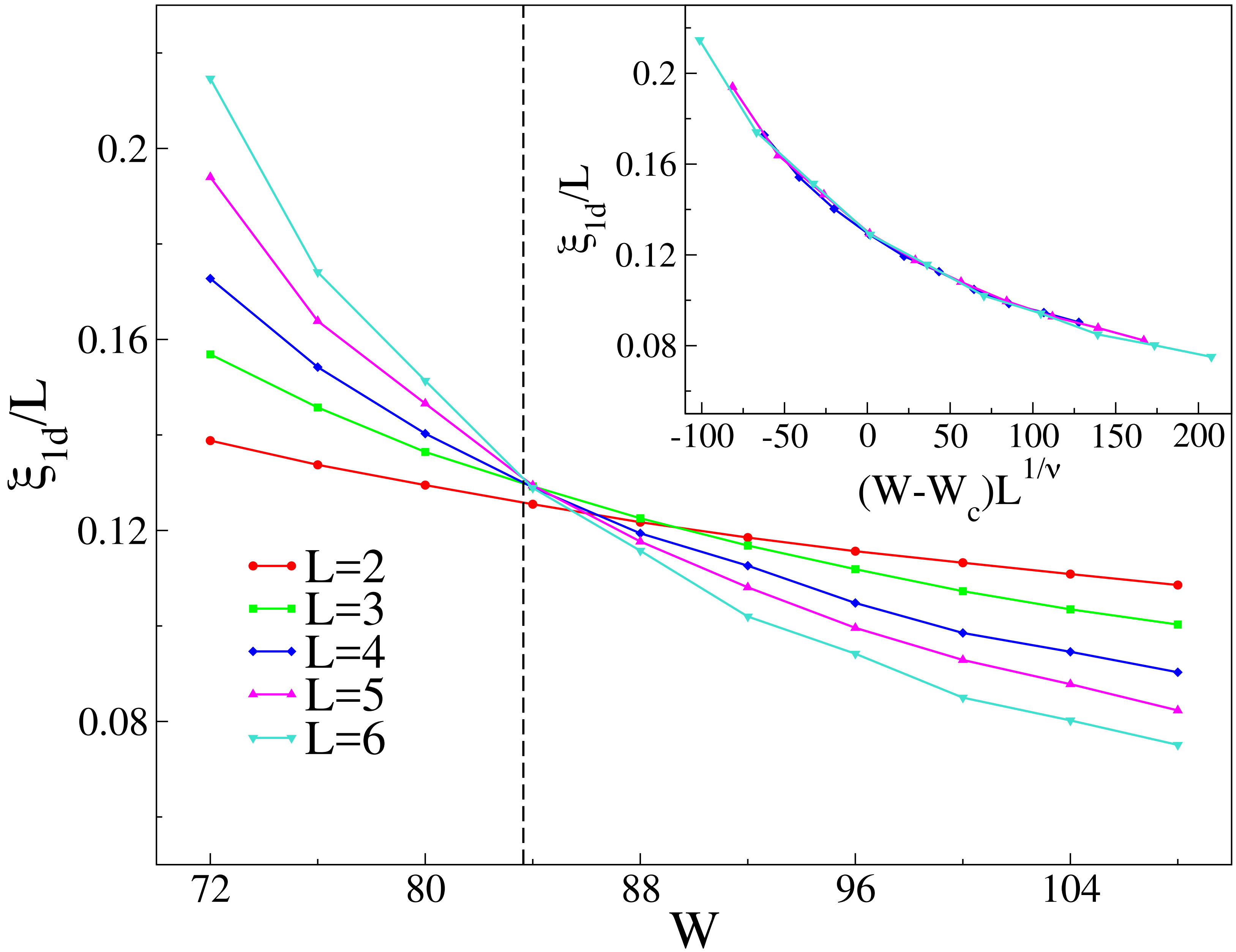


$$\langle \mathrm{Im} G^{(x)} \rangle \sim \exp[-x/\xi_{1d}]$$

Dimensionless quasi-$1d$ localization length

$$\frac{\xi_{1d}}{L} \sim \begin{cases} (L/\xi)^{d-2} & \text{for } W < W_c \\ \text{cst} & \text{for } W = W_c \\ \xi/L & \text{for } W > W_c \end{cases}$$

Finite size scaling in $d = 6$



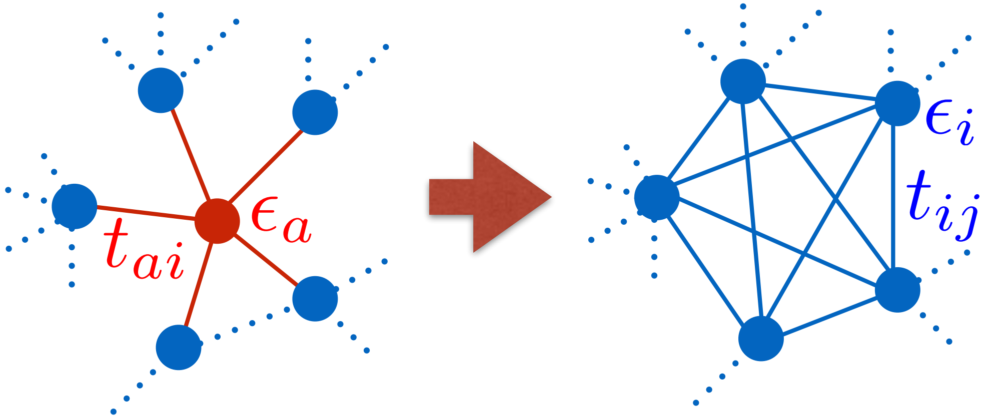
$\nu \simeq 0.84$

$W_c \simeq 83.8$

$a \simeq -1$

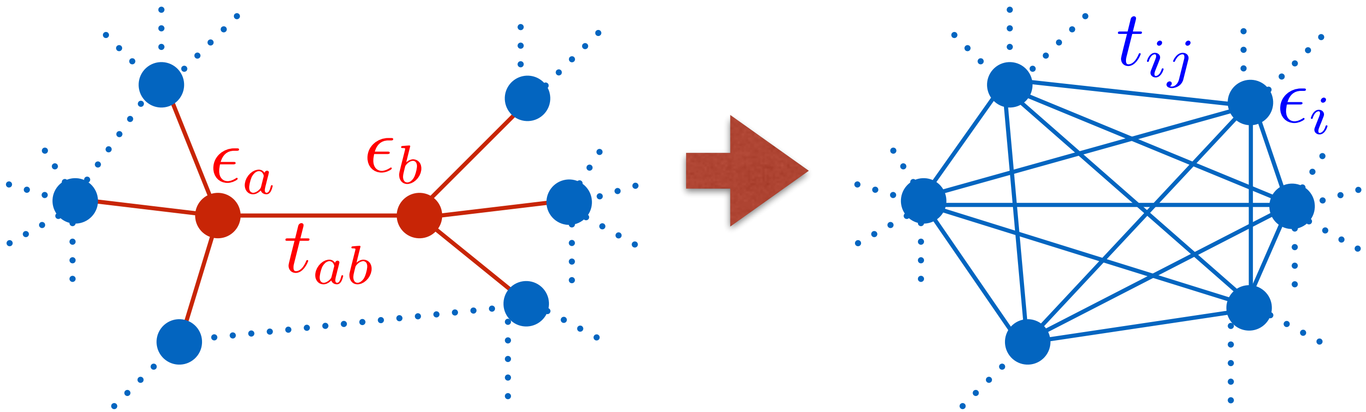
Strong Disorder RG

Integrate out iteratively the strongest energy scale



$$\epsilon_i \rightarrow \epsilon_i - \frac{t_{ai}^2}{\epsilon_a}$$

$$t_{ij} \rightarrow t_{ij} - \frac{t_{ai}t_{aj}}{\epsilon_a}$$



Exact RG transformations

Aoki '80

$$\epsilon_i \rightarrow \epsilon_i - \frac{\epsilon_b t_{ai}^2 - 2t_{ab}t_{ai}t_{bi} + \epsilon_a t_{bi}^2}{\epsilon_a \epsilon_b - t_{ab}^2}$$

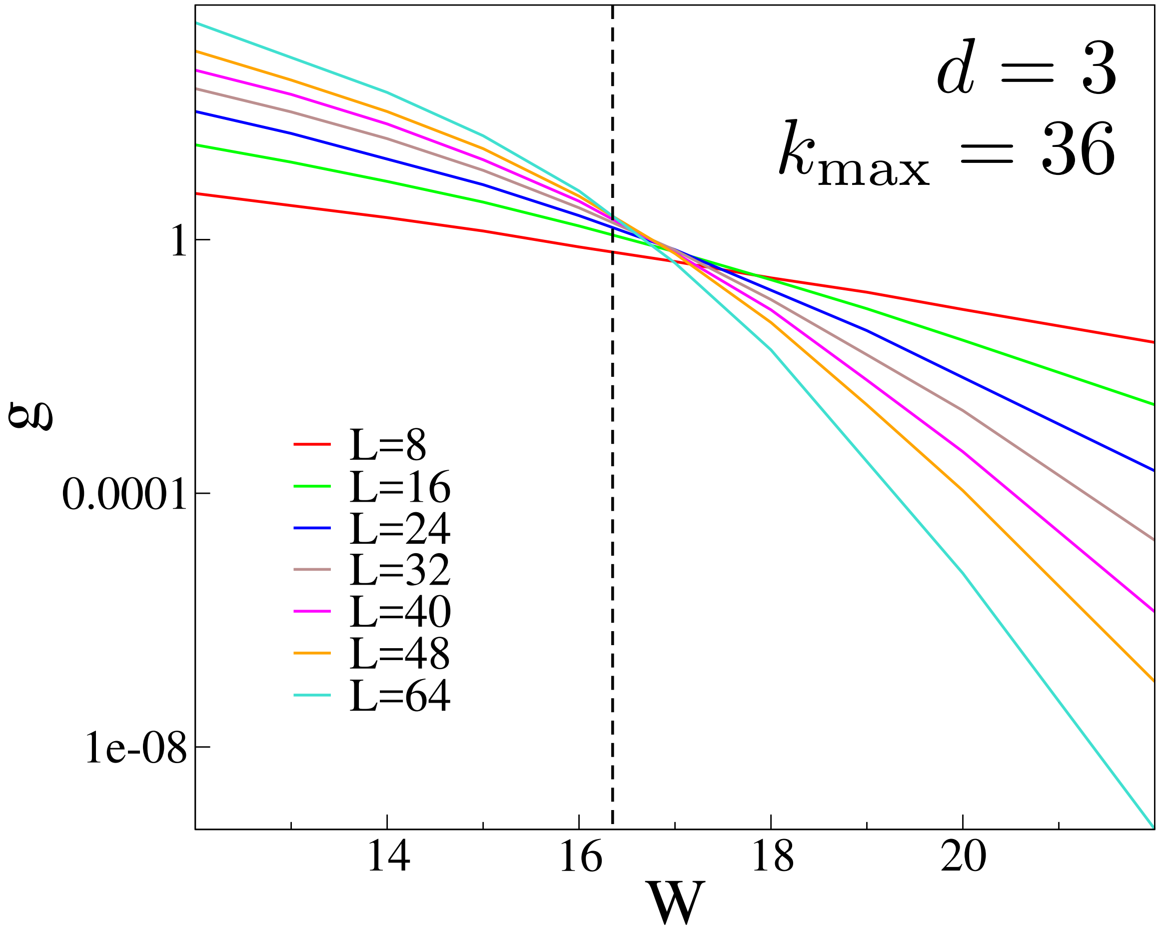
$$t_{ij} \rightarrow t_{ij} - \frac{\epsilon_b t_{ai}t_{aj} - t_{ab}(t_{ai}t_{bj} + t_{bi}t_{aj}) + \epsilon_a t_{bi}t_{bj}}{\epsilon_a \epsilon_b - t_{ab}^2}$$

Eliminating weak (irrelevant?) terms

Removing weak couplings ➡ Maximum coordination number $k_{\max}$

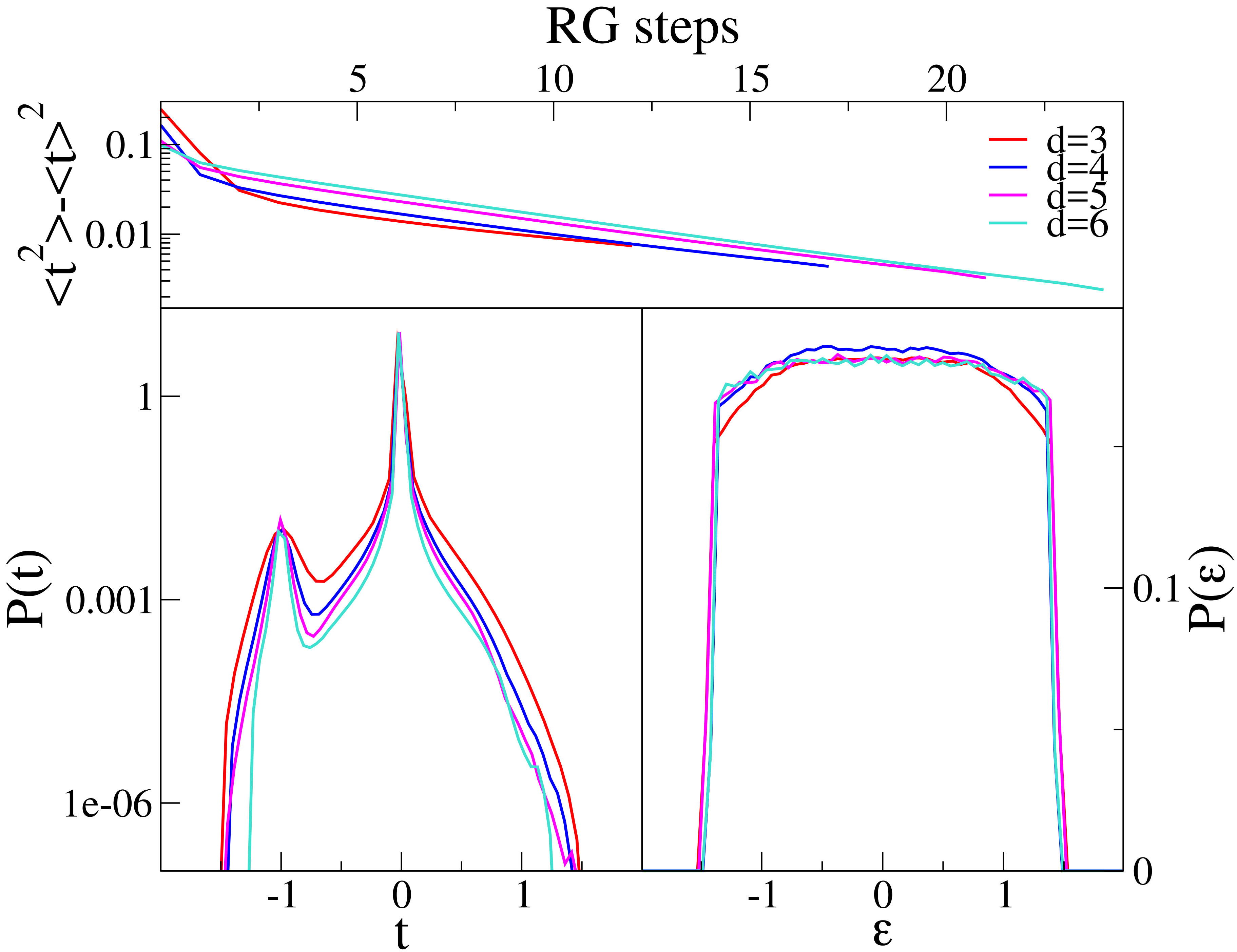
Fisher '92; Iglói & Monthus '05; Kováks & Iglói '11; Monthus & Garel '09; Mard & al '14; Mard & al '15

Computation time $\sim L^{2d} k_{\max}^3 \longrightarrow$ Access much bigger sizes

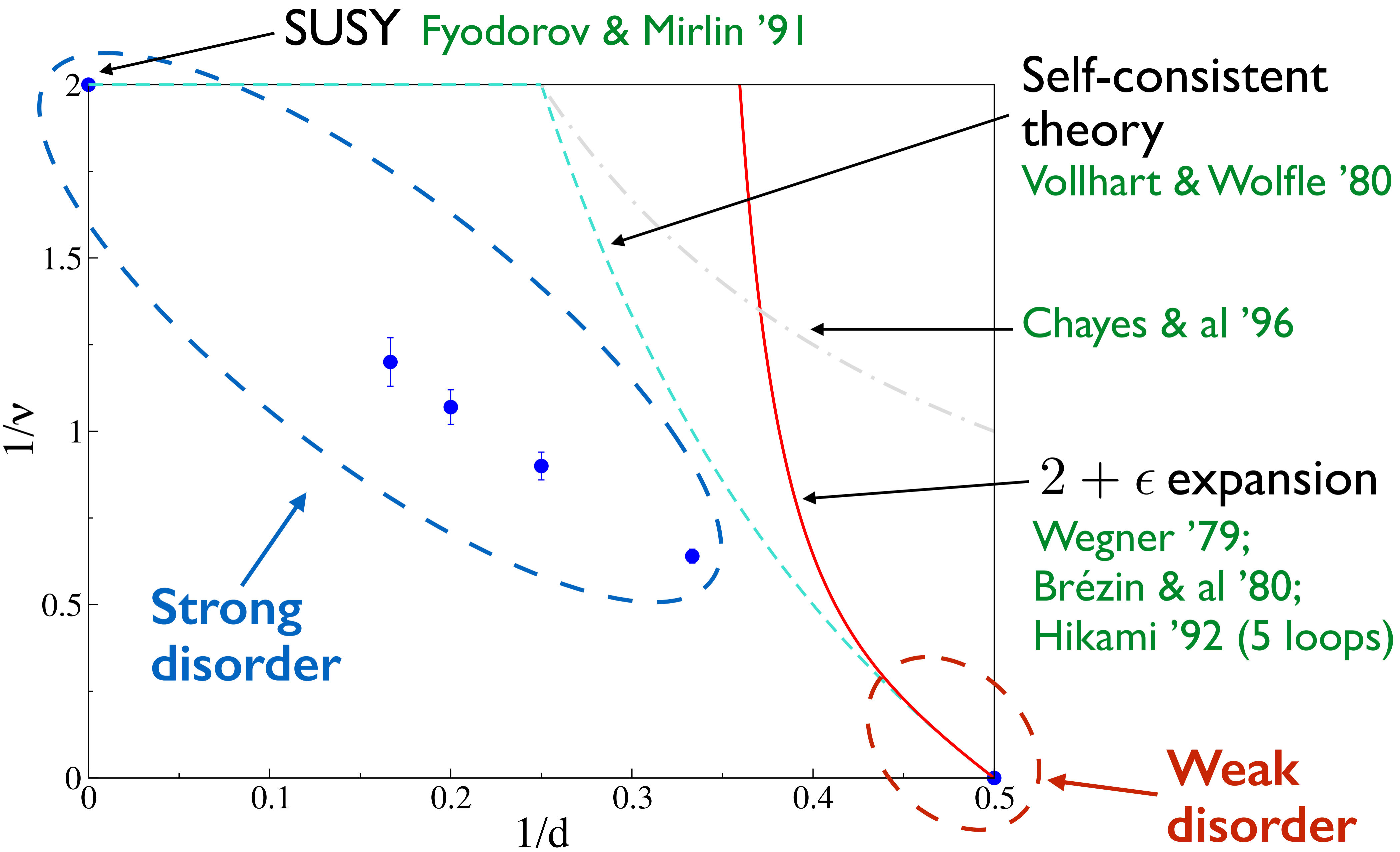


Excellent agreement with numerical results in all dimensions!

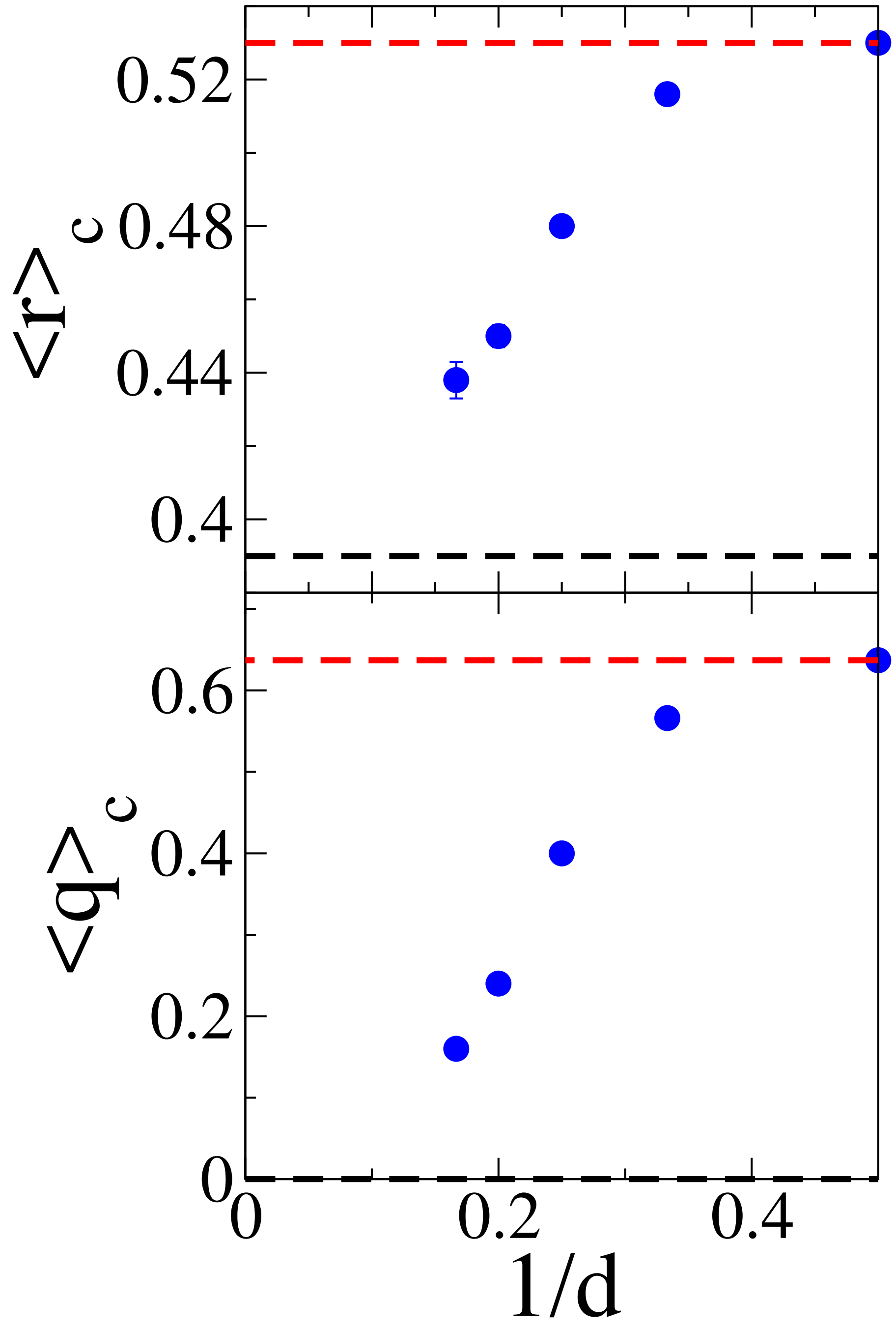
Flow of the SDRG at W_c

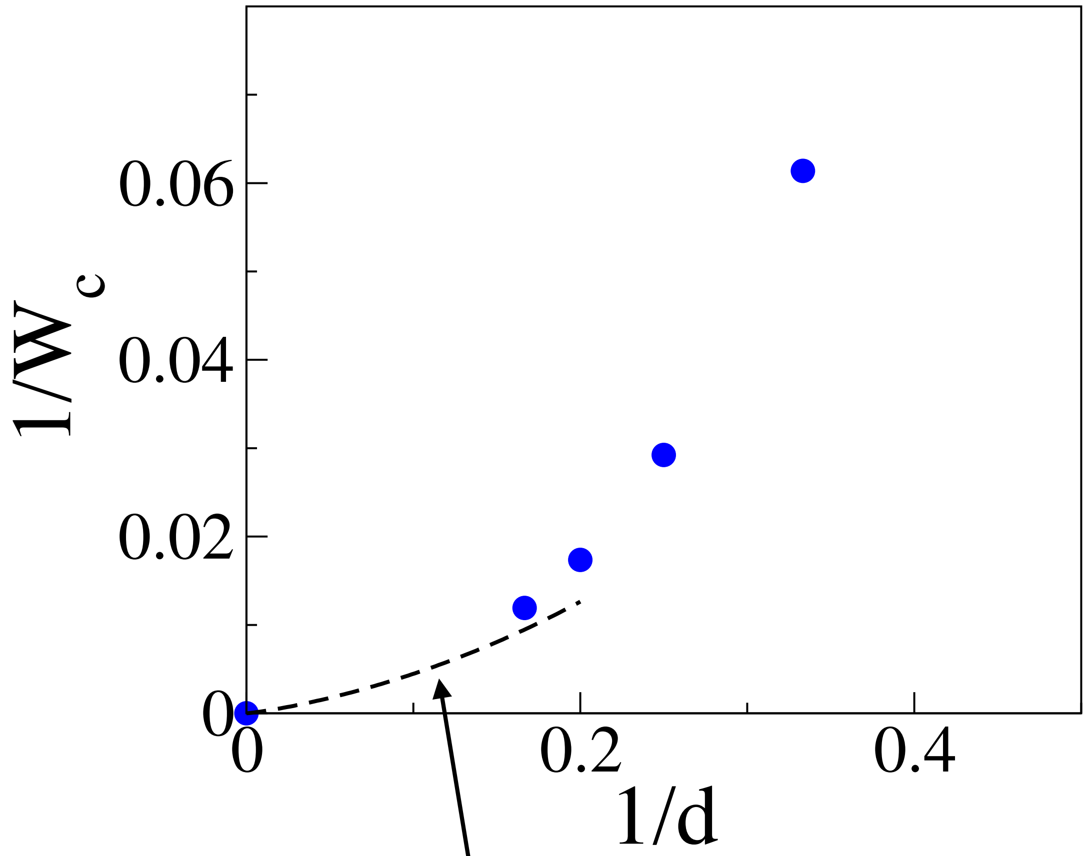


Critical exponent ν

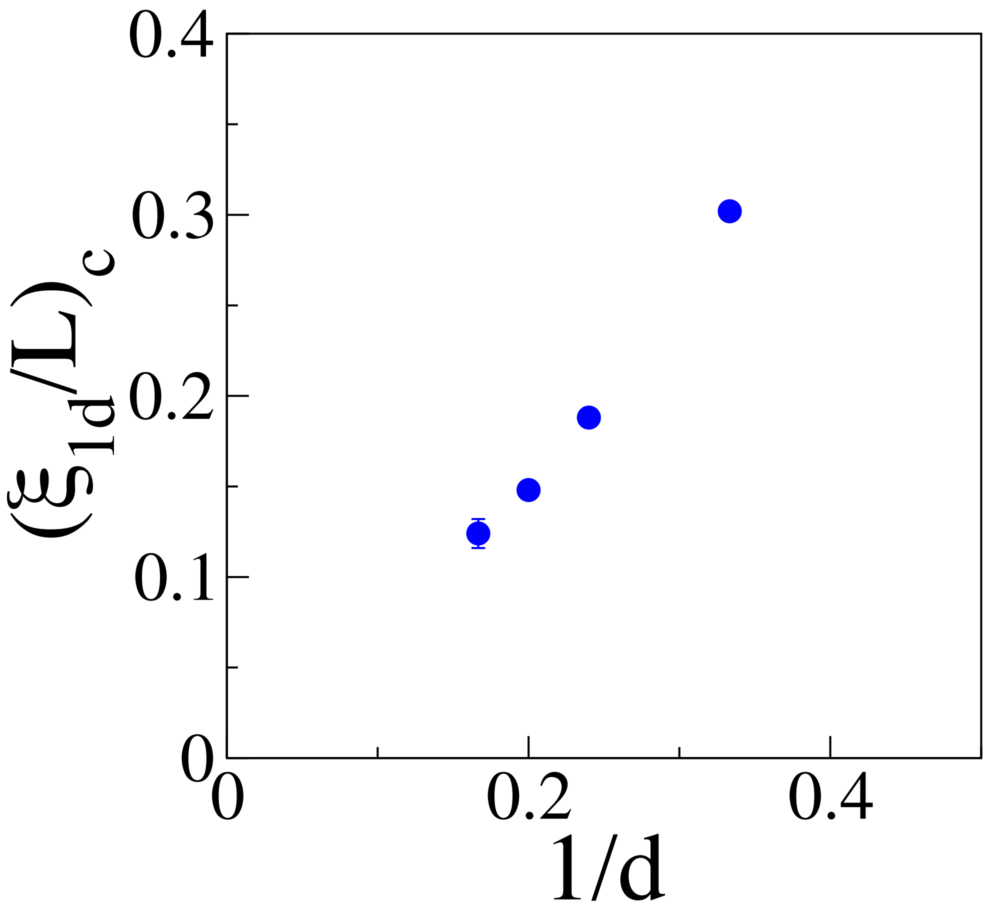


Critical values (1)

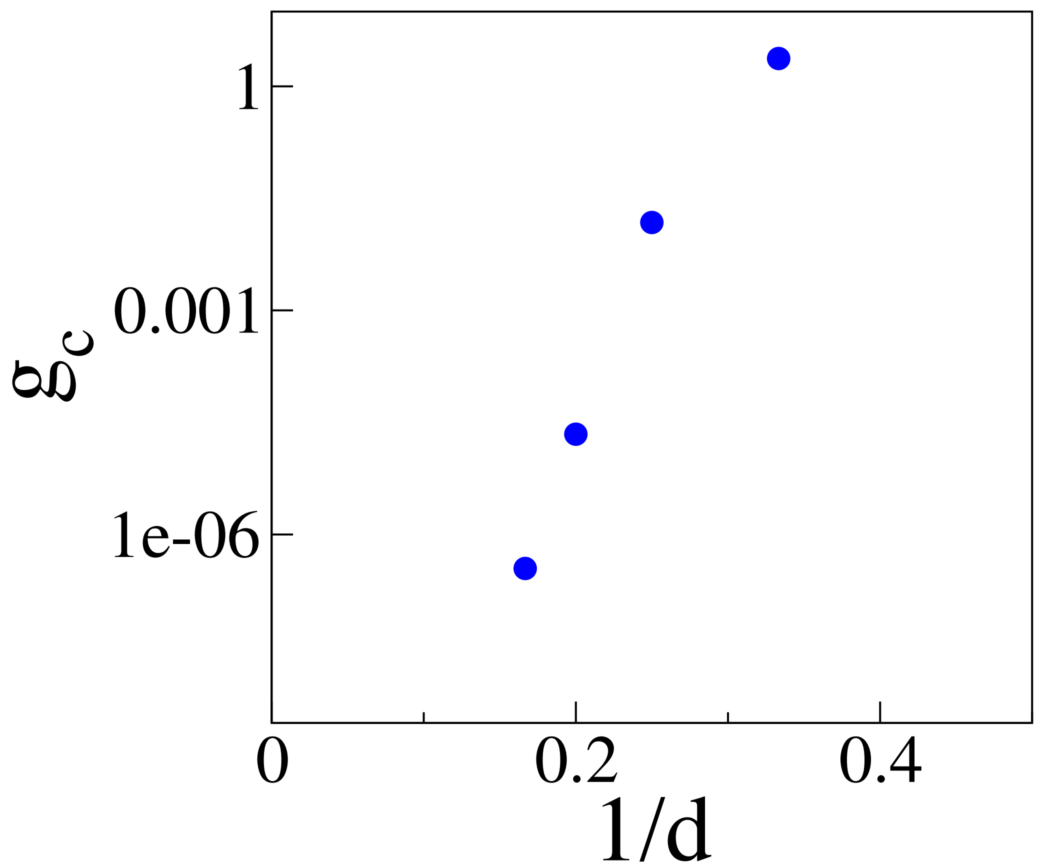




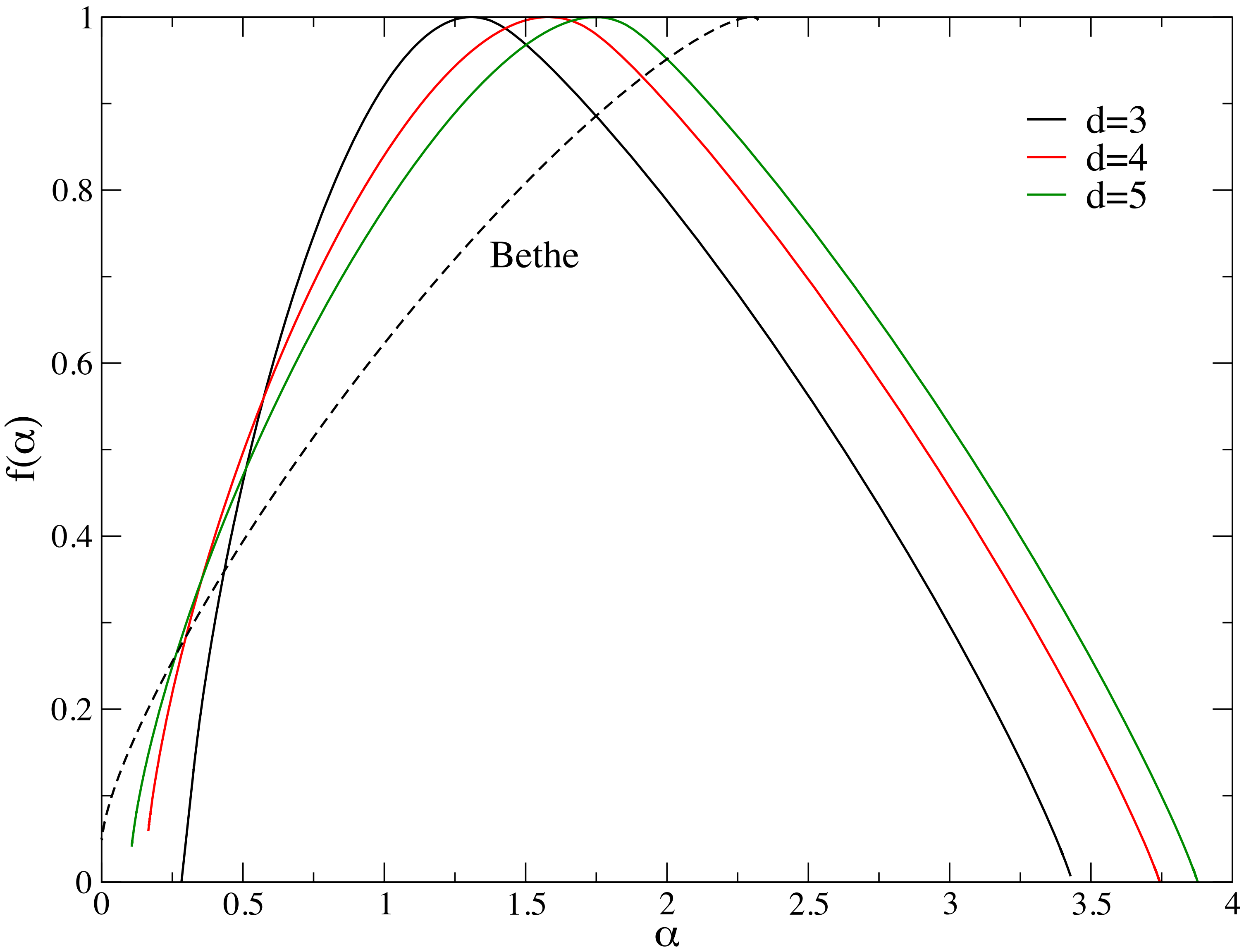
Critical values (2)



The critical values of all observables gradually approach the ones of the localized phase



Multifractality



The multifractal specta at the AL broadens as d is increased and seems to approach smoothly the one of the Bethe lattice

Summary of the results

- **No finite upper critical dimension**
- **AL in high d is controlled by Strong Disorder**
  Infinite Randomness fixed point in $d \to \infty$? Fisher '92
- **No evidence of the intermediate phase in large dimensions** (transitions of level statistics, IPR, conductivity, $\ldots$, all occur at the same point up to $d = 6$)
- **Dramatic finite size effects found on the Bethe lattice are explained by:**
  - **"Quasi localized" character** of the AT fixed point in $d \to \infty$ (the critical values are the same as in the localized phase)
  - Exponent a describing finite size corrections is approximatively constant ($a \simeq -1$ in all dimensions up to 6)
    Logarithmic corrections $\propto 1/\log N$ for $d \to \infty$

Perspectives

- **Study higher dimensions via SDRG** (in progress)
  Preliminary results up to $d = 10$ Mard & al '15

- **Implementation of a real space RG introduced for the Anderson models with long range hopping**
  (in progress) Levitov '90; Mirlin & Evers '00

- $1/d$ **expansion of the replicated fully-connected NL**σ**M**
  Numerical data suggest that convergence in $1/d$ is smooth

- **Non-perturbative (functional?) Renormalization Group approach for the Anderson Model** (in progress)
  From weak disorder to strong disorder