Ising n-fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity

November 26th, 2012

A. Bostan[¶], S. Boukraa[||], G. Christol[‡], S. Hassani[§], J-M. Maillard[£]

[¶] INRIA, Bâtiment Alan Turing, 1 rue Honoré d'Estienne d'Orves, Campus de l'École Polytechnique, 91120 Palaiseau, France

[||] LPTHIRM and Département d'Aéronautique, Université de Blida, Algeria

[‡] Institut de Mathématiques de Jussieu, UPMC, Tour 25, 4ème étage, 4 Place Jussieu, 75252 Paris Cedex 05, France

[§] Centre de Recherche Nucléaire d'Alger, 2 Bd. Frantz Fanon, B.P. 399, 16000 Alger, Algeria

[£] LPTMC, UMR 7600 CNRS, Université de Paris 6, Tour 23, 5ème étage, case 121, 4 Place Jussieu, 75252 Paris Cedex 05, France

Abstract.

We show that the n-fold integrals $\chi^{(n)}$ of the magnetic susceptibility of the Ising model, as well as various other n-fold integrals of the "Ising class", or n-fold integrals from enumerative combinatorics, like lattice Green functions, are actually *diagonals of rational functions*. As a consequence, the power series expansions of these solutions of linear differential equations "Derived From Geometry" are *globally bounded*, which means that, after just one rescaling of the expansion variable, they can be cast into series expansions with *integer coefficients*. Besides, in a more enumerative combinatorics context, we show that generating functions whose coefficients are expressed in terms of nested sums of products of binomial terms can also be shown to be *diagonals of rational functions*. We give a large set of results illustrating the fact that the unique analytical solution of Calabi-Yau ODEs, and more generally of MUM ODEs, is, almost always, diagonal of rational functions. We revisit Christol's conjecture that globally bounded series of G-operators are necessarily diagonals of rational functions. We provide a large set of examples of globally bounded series, or series with integer coefficients, associated with modular forms, or Hadamard product of modular forms, or associated with Calabi-Yau ODEs, underlying the concept of modularity. We finally address the question of the relations between the notion of *integrality* (series with integer coefficients, or, more generally, globally bounded series) and the *modularity* (in particular integrality of the Taylor coefficients of mirror map), introducing new representations of Yukawa couplings.

PACS: 05.50.+q, 05.10.-a, 02.30.Hq, 02.30.Gp, 02.40.Xx
AMS Classification scheme numbers: 34M55, 47E05, 81Qxx, 32G34, 34Lxx, 34Mxx, 14Kxx

Key-words: Diagonals of rational functions, Hadamard products, Hurwitz products, series with integer coefficients, globally bounded series, G-series, G-

operators, differential equations Derived From Geometry, elliptic curves, elliptic integrals, nome, Hauptmoduls, modular forms, Calabi-Yau ODEs, modularity, modular polynomial, modular equation, modularity conjecture, mirror maps, Yukawa coupling, Schwarzian derivatives, lattice Green functions, MUM linear ODEs, Picard-Fuchs systems, Gauss-Manin connections, embedded resolution of singularities, Mahler measures.

1. Introduction

The series expansions of many magnetic susceptibilities (or many other quantities, like the spontaneous magnetisation) of the Ising model on various lattices in arbitrary dimensions are actually series with *integer coefficients* [1, 2, 3]. This is a consequence of the fact that, in a van der Waerden type expansion of the susceptibility, all the contributing graphs are the ones with exactly two odd-degree vertices and the number of such graphs is an integer. When series expansions in theoretical physics, or mathematical physics, do not have such an obvious counting interpretation, the puzzling emergence of series with *integer coefficients* is a strong indication that some fundamental structure, symmetry, concept have been overlooked, and that a deeper understanding of the problem remains to be discovered‡. Algebraic functions are known to produce series with *integer coefficients*. Eisenstein's theorem [6, 7] states that the Taylor series of a (branch of an) algebraic function can be recast into a series with integer coefficients, up to a rescaling by a constant (Eisenstein constant). An intriguing result due to Fatou [8] (see pp. 368–373) states that a power series with *integer* coefficients and radius of convergence (at least) 1, is either rational, or transcendental. This result also appears in Pólya and Szegö's famous Aufgaben book [9] (see Problem VIII-167). Pólya [10] conjectured a stronger result, namely that a power series with integer coefficients which converges in the open unit disk is either rational, or admits the *unit circle as a natural boundary* (i.e. it has no analytic continuation beyond the unit disk). This was eventually proved¶ by Carlson [12, 13]. Along this natural boundary line, it is worth recalling [14, 15, 16, 17, 18] that the series expansions of the full magnetic susceptibility of the 2D Ising model corresponds to a power series with integer coefficients†. For them, a unit circle natural boundary certainly arises [19] (with respect to the modulus variable k), but, unfortunately, this cannot be justified by Carlson's theorem††.

A series with natural boundaries *cannot be D-finite*, i.e. solution of a linear differential equation with polynomial coefficients [20, 21]♯. For simplicity, let us restrict to series with *integer coefficients* (or series that have integer coefficients up to a variable rescaling), that are series expansions of D-finite functions. Wu, McCoy, Tracy and Barouch [25] have shown that the previous full magnetic susceptibility of the 2D Ising model can be expressed (up to a normalisation factor $(1-s)^{1/4}/s$, see [16, 26]) as an infinite sum of n-fold integrals, denoted by $\tilde{\chi}^{(n)}$, which are *actually*

‡ The emergence of *positive integer* coefficients corresponds to the existence of some underlying measure [4] (see also the concept of Mahler measures [5]).

¶ The Pólya-Carlson result can be used to prove that some integer sequences, such as the sequence of prime numbers (p_n) [11], do not satisfy any linear recurrence relation with polynomial coefficients.

† In some variable w [15, 14, 16, 17]. In the modulus variable k, one needs to perform a simple rescaling by a factor 2 or 4 according to the type of (high, or low temperature) expansions.

†† The radius of convergence is 1 with respect to the modulus variable k, in which the series *does not have* integer coefficients, being *globally bounded* only (this means that it can be recast into a series with integer coefficients by one rescaling of the variable k). If one considers the series expansion with respect to another variable (such as w) in which the series *does have* integer coefficients, then the radius of convergence is not 1.

♯ D-finite series are sometimes called *holonomic*. A priori, these notions differ: a function $f(x_1, \ldots, x_r)$ is called *D-finite* if all its partial derivatives $D_1^{n_1} \cdots D_r^{n_r} \cdot f$ generate a finite dimensional space over $\mathbb{Q}(x_1, \ldots, x_r)$, and *holonomic* if the functions $x_1^{\alpha_1} \cdots x_r^{\alpha_r} D_1^{\beta_1} \cdots D_r^{\beta_r} \cdot f$ obtained by multiplying monomials in the variables and higher-order derivatives of f subject to the constraint $\alpha_1 + \cdots + \alpha_r + \beta_1 + \cdots + \beta_r \leq N$ span a vector space whose dimension over $\mathbb{Q}$ grows like $O(N^r)$. The equivalence of these notions is proved by profound results of Bernšteĭn [22] and Kashiwara [23, 24].

D-finite♯. We found out that the corresponding (minimal order) differential operators are Fuchsian [14, 16], and, in fact, "special" Fuchsian operators: the critical exponents for *all* their singularities are *rational numbers*, and their Wronskians are N-th roots of *rational functions* [27]. Furthermore, it has been shown later that these $\tilde{\chi}^{(n)}$'s are, in fact, solutions of *globally nilpotent* operators [28], or G-operators [29, 30]. It is worth noting that the series expansions, at the origin, of the $\tilde{\chi}^{(n)}$'s, in a well-suited variable [16, 26] w, actually have *integer coefficients*, even if this result does not have an immediate proof† for all integers n (in contrast with the full susceptibility).

From the first truncated series expansions of $\tilde{\chi}^{(n)}$, the coefficients for generic n can be inferred [28]

$$\begin{aligned}\tilde{\chi}^{(n)}(w) \;\; = \;\; & 2^n \cdot w^{n^2} \cdot \Big(1 \, + 4\,n^2 \cdot w^2 \, + 2 \cdot (4\,n^4 \, + 13\,n^2 \, + 1) \cdot w^4 \\ & + \frac{8}{3} \cdot (n^2+4)\,(4\,n^4 \, + 23\,n^2 + 3) \cdot w^6 \\ & + \frac{1}{3} \cdot \, (32\,n^8 \, + 624\,n^6 \, + 4006\,n^4 \, + 8643\,n^2 \, + 1404) \cdot w^8 \\ & + \frac{4}{15} \cdot \, (n^2+8) \cdot \, (32\,n^8 \, + 784\,n^6 \, + 6238\,n^4 \, + 16271\,n^2 \, + 3180) \cdot w^{10} \, + \, \cdots \Big).\end{aligned} \tag{1}$$

Note that the coefficients of the expansion of $\tilde{\chi}^{(n)}(w)/2^n$ depend on n^2. Note that these coefficients are *integer coefficients* when n is *any integer*, this integrality property [31] of the coefficients becoming straightforward to see when one remarks that $(4\,n^4 \, + 23\,n^2 + 3)$ and $(32\,n^8 \, + 624\,n^6 \, + 4006\,n^4 \, + 8643\,n^2 \, + 1404)$ are of the form $n \cdot (n^2-1)\,f(n) + \, 3\,g(n)$ (respectively $f(n) = \, 4\,n$ and $g(n) \, = \, 9\,n^2 \, + 1$, and $f(n) = \, 2\,n \cdot \, (16\,n^4 \, + 328\,n^2 \, + 2331)$ and $g(n) \, = \, 4435\,n^2 \, + 468)$, and, hence, are always divisible by 3, that $(32\,n^8 \, + 784\,n^6 \, + 6238\,n^4 \, + 16271\,n^2 \, + 3180)$ is of the form $n \cdot \, (n^2-1)\,(n^2-4) \cdot \, f(n) \, + 3 \cdot \, 5 \cdot g(n)$, with $f(n) \, = \, 16\,n \cdot \, (2\,n^2+59)$ and $g(n) \, = \, 722\,n^4 \, + 833\,n^2 \, + 212$, hence, always divisible by 3 and 5.

These coefficients are valid up to w^2 for $n \, \geq \, 3$, w^4 for $n \, \geq \, 5$, w^6 for $n \, \geq \, 7$, w^8 for $n \geq 9$, and w^{10} for $n \geq 11$ (in particular it should be noted that $\tilde{\chi}^{(n)}$ is an even function of w only for even n). Further studies on these $\tilde{\chi}^{(n)}$'s showed the fundamental role played by the theory of elliptic functions¶ (elliptic integrals, *modular forms*) and, much more unexpectedly, *Calabi-Yau ODEs* [32, 33]. These recent structure results thus suggest to see the occurrence of series with integer coefficients as a consequence of *modularity* [34] (modular forms, mirror maps [32, 33, 34, 35], etc) in the Ising model.

Along this line, many other examples of series with *integer coefficients* emerged in mathematical physics (differential geometry, lattice statistical physics, enumerative combinatorics, *replicable functions*♯ ...). One must, of course, also recall Apéry's results [44]. We give, in Appendix A, a list of *modular forms*, and their associated series with integer coefficients, corresponding to various lattice Green functions [45,

♯ For Ising models on higher dimensional lattices [1, 2, 3] no such decomposition of susceptibilities, as an infinite sum of D-finite functions, should be expected at first sight.

† We are interested in this paper in the emergence of integers as coefficients of D-finite series. In general, this emergence is not obvious: it cannot be simply explained at the level of the linear recurrence satisfied by the coefficients, as illustrated by the case of Apéry's calculations (see also Appendix G below).

¶ Which is not a surprise for Yang-Baxter integrability specialists.

♯ The concept of replicable functions [36] is closely related to *modular functions* [37, 38], (see the replicability of Hauptmoduls [38]), Calabi-Yau threefolds, and more generally the concept of *modularity* [34, 39, 40, 41, 42, 43] (the third étale cohomology of a rigid Calabi-Yau threefold comes from a modular form of weight 4, ...).

46, 47, 48].This integrality is also seen in the *nome* and in other quantities like the *Yukawa coupling* [32].

We restrict to series with integer coefficients, or, more generally, *globally bounded* [49] series of *one complex variable*, but it is clear that this integrality property does also occur in physics with *several complex variables*: they can, for instance, be seen for the previous (D-finite§) n-fold integrals $\tilde{\chi}^{(n)}$ for the anisotropic Ising model [50] (or for the Ising model on the checkerboard lattice), or on the example of the lattice Ising models with a magnetic field‡ (see for instance, Bessis et al. [4]).

One purpose of this paper is to "disentangle" the notion of series with integer coefficients (*integrality*) and the notion of *modularity* [34, 39, 40, 41, 42, 43]. In this down-to-earth paper we will use the wording of "modularity", not to refer to the *modularity conjecture* and other Serre's results that certain Geometric Galois representations are modular, but as a quick proxy word to say that a series solution of a linear differential operator *as well* as the nome, and hopefully other series (Yukawa coupling, ...) *are all series with integer coefficients*.

We will show that the $\tilde{\chi}^{(n)}$'s are *globally bounded* series, as a consequence of the fact that they actually are *diagonals of rational functions for any value of the integer* n. We will generalise these ideas, and show that an extremely large class of problems of mathematical physics can be interpreted in terms of *diagonal of rational functions*: n-fold integrals with algebraic integrand of a certain type that we will characterise, Calabi-Yau ODEs, MUM linear ODEs [52], series whose coefficients are *nested sums of binomials*, etc. We take, here, a learn-by-example approach: on such questions one gets a much deeper understanding from highly non-trivial examples than from general mathematical demonstrations [53, 54].

2. Series integrality

2.1. Globally bounded series

Let us recall the definition of being *globally bounded* [49] for a series. Consider a series expansion with rational coefficients, with non-zero radius of convergence†. The series is said to be globally bounded if there exists an integer N such that the series can be recast into a series with integer coefficients with just one rescaling $x \rightarrow N\,x$.

A necessary condition for being globally bounded is that only a finite number of primes occurs for the factors of the denominators of the rational number series coefficients. There is also a condition on the growth of these denominators, that must be bounded exponentially [49], in such a way that the series has a non-zero p-adic radius of convergence for all primes p.

When this is the case, it is easy to see that these series can be recast, with just one rescaling, into series with *integer coefficients*¶.

It will be seen, in a forthcoming section (see (3) below), that the series expansion of *diagonals of rational functions* [55, 56, 57] *are necessarily globally bounded*.

§ For several complex variables the ODEs of the paper are replaced by Picard-Fuchs systems.
‡ Along this line original alternative representations of the partition function of the Ising model in a magnetic field are also worth recalling [51].
† A series like the Euler-series $\sum_{n=0}^{\infty} n! \cdot x^n$ which has integer coefficients is excluded.
¶ For a first set of series with integer coefficients, see Appendix A, where a set of such series with integer coefficients corresponding to *modular forms* is displayed.

2.2. Globally logarithmically bounded series

There is another notion, weaker than being globally bounded, namely the notion of being *globally logarithmically*§ *bounded*.

As an example consider the series expansion of ${}_2F_1([1/4, 1/2], [5/4], 4\,x)$. This series is *not globally bounded*

$$ {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right], 4\,x\right) \quad = \quad 1 \; + \frac{2}{5}\,x \; + \frac{2}{3}\,x^2 \, + \frac{20}{13}\,x^3 \, + \frac{70}{17}\,x^4 \, + 12\,x^5 $$
$$ + \, \frac{924}{25}\,x^6 \, + \frac{3432}{29}\,x^7 \, + 390\,x^8 \, + \frac{48620}{37}\,x^9 \quad + \; \cdots \qquad (2) $$

When looking at the denominators of the series coefficients, one finds that *almost all primes* of the form $4\,\ell + 1$ occur. There is no way to recast this series into a series with integer coefficients with one rescaling.

Let us denote $\theta \, = \, x \cdot d/dx$. The hypergeometric function ${}_2F_1([1/4, \, 1/2], [5/4], x)$ is solution of the operator

$$ \omega \quad = \quad \theta \cdot \Bigl(\theta \, + \, \frac{1}{4}\Bigr) \quad - \, x \cdot \, \Bigl(\theta \, + \, \frac{1}{4}\Bigr) \cdot \, \Bigl(\theta \, + \, \frac{1}{2}\Bigr), \qquad (3) $$

which clearly factors† $4\,\theta \, + \, 1$ at the right:

$$ \Bigl(2\,\theta - x \, \cdot \, (2\,\theta \, + \, 1)\Bigr) \cdot \, (4\,\theta \, + \, 1). \qquad (4) $$

Consequently, the action of $4\,\theta + 1$ on ${}_2F_1([1/4, 1/2], [5/4], x)$ becomes the solution of the order-one globally nilpotent operator $2\,\theta - x \cdot (2\,\theta + 1)$, and is, thus, an *algebraic function* (rational or N-th root of rational), namely $(1-x)^{-1/2}$. The hypergeometric function ${}_2F_1([1/4, 1/2], [5/4], x)$ is *not* globally bounded. We will see below that, consequently, it *cannot be the diagonal of a rational function*, however it is not a "wild" series, the denominators do not grow "too fast": it is actually such that a simple order-one operator, namely $4\,\theta \, + \, 1$, acting on this series, changes it into a *diagonal of rational function*. Other examples of globally logarithmically bounded hypergeometric series are given in Appendix B.

3. Minimal recalls on diagonals of rational functions

Let us recall here the concept of *diagonal* of a "function"¶, and some of its most important properties.

3.1. Definition of the diagonal of a rational function

Assume that $\mathcal{F}(z_1, \ldots, z_n) \; = \; P(z_1, \ldots, z_n)/Q(z_1, \ldots, z_n)$ is a rational function, where P and Q are polynomials with *rational coefficients* such that $Q(0, \ldots, 0) \neq 0$. This assumption implies that $\mathcal{F}$ can be expanded as a Taylor series at the origin

$$ \mathcal{F}\Bigl(z_1, \, z_2, \, \ldots, \, z_n\Bigr) \quad = \qquad (5) $$
$$ \sum_{m_1\,=\,0}^{\infty} \, \sum_{m_2\,=\,0}^{\infty} \, \cdots \, \sum_{m_n\,=\,0}^{\infty} \, F_{m_1,\, m_2,\, \ldots,\, m_n} \cdot \, z_1^{m_1} \, z_2^{m_2} \, \cdots \, z_n^{m_n} \quad \in \quad \mathbb{Q}[[z_1, \ldots, z_n]]. $$

§ For a series $\sum a_n\,x^n$ being "bounded" means bounded by 1, p-adically: $|a_n|_p \, \leq \, 1$, i.e. a_n has no p factor at the denominator.) Being logarithmically bounded means "with logarithmic grows", i.e. $|a_n|_p \, \leq \, n$.

† This result can, of course, straightforwardly be generalised to ${}_2F_1([a, \, b], [1 + a], x)$.

¶ This is an abuse of language: the "functions" are in fact defined by *series* of several complex variables: they have to be analytical (no Puiseux series).

The *diagonal of* $\mathcal{F}$ is defined as the series of *one variable*

$$\mathsf{Diag}\Big(\mathcal{F}\Big(z_1,\, z_2,\, \ldots,\, z_n\Big)\Big) \quad = \quad \sum_{m\,=\,0}^{\infty} F_{m,\, m,\, \ldots,\, m} \cdot z^m \quad \in \quad \mathbb{Q}[[z]]. \tag{6}$$

More generally, one can define, in a similar way, the diagonal of any multivariate power series $\mathcal{F} \in K[[z_1, \ldots, z_n]]$, with coefficients in an arbitrary field K (possibly of positive characteristic)‡.

3.2. Main properties of diagonals

The concept of diagonal of a function has a lot of interesting properties (see for instance [59]). Let us recall, through examples, some of the most important ones.

The study of diagonals goes back, at least, to Pólya [60], in a combinatorial context, and to Cameron and Martin [61] in an analytical context *related to Hadamard products* [62]. Pólya showed that the diagonal of a rational function in *two variables* is always an *algebraic function*. The most basic example is $\mathcal{F} = 1/(1 - z_1 - z_2)$, for which

$$\mathsf{Diag}(\mathcal{F}) \quad = \quad \mathsf{Diag}\left(\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\binom{m_1+m_2}{m_1} \cdot z_1^{m_1} z_2^{m_2}\right)$$
$$= \quad \sum_{m=0}^{\infty}\binom{2m}{m} \cdot z^m \quad = \quad \frac{1}{\sqrt{1-4z}}. \tag{7}$$

The proof of Pólya's result is based on the simple observation that the diagonal $\mathsf{Diag}(\mathcal{F})$ is equal to the coefficient of z_1^0 in the expansion of $\mathcal{F}(z_1, z/z_1)$. Therefore, by Cauchy's integral theorem, $\mathsf{Diag}(\mathcal{F})$ is given by the contour integral

$$\mathsf{Diag}(\mathcal{F}) \quad = \quad [z_1^{-1}]\, \mathcal{F}(z_1, z/z_1)/z_1 \quad = \quad \frac{1}{2\pi i}\oint_{\gamma} \mathcal{F}(z_1, z/z_1)\, \frac{dz_1}{z_1}, \tag{8}$$

where $[z_1^n]$ means† extracting the n-th coefficient of a power series, and where the contour γ is a small circle around the origin. Therefore, by Cauchy's residue theorem, $\mathsf{Diag}(\mathcal{F})$ is the sum of the residues of the rational function $\mathcal{G} = \mathcal{F}(z_1, z/z_1)/z_1$ at all its singularities $s(z)$ with zero limit at $z = 0$. Since the residues of a rational function of two variables are algebraic functions, $\mathsf{Diag}(\mathcal{F})$ is itself an algebraic function.

For instance, when $\mathcal{F} = 1/(1 - z_1 - z_2)$, then $\mathcal{G} = \mathcal{F}(z_1, z/z_1)/z_1$ has two poles at $s = \frac{1}{2}(1 \pm \sqrt{1-4z})$. The only one approaching zero when $z \to 0$ is $s_0 = \frac{1}{2}(1 - \sqrt{1-4z})$. If $p(s)/q(s)$ has a simple pole at s_0, then its residue at s_0 is $p(s_0)/q'(s_0)$. Therefore

$$\mathsf{Diag}(\mathcal{F}) \quad = \quad \frac{1}{2\pi i}\oint_{\gamma} \frac{dz_1}{z_1 - z_1^2 - z}$$
$$= \quad \mathsf{Res}_{s_0}\, \frac{dz_1}{z_1 - z_1^2 - z} \quad = \quad \frac{1}{1-2s_0} \quad = \quad \frac{1}{\sqrt{1-4z}}. \tag{9}$$

When passing from two to more variables, diagonalisation may still be interpreted using contour integration of a multiple complex integral over a so-called *vanishing*

‡ The definition even extends to multivariate Laurent power series, see e.g. [58].
† This is a convenient notation, very often used in combinatorics [63].

cycle [64]. However, the result *is not* an algebraic function anymore. A simple example is $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1 z_2 - z_1 z_3)$, for which

$$\mathsf{Diag}(\mathcal{F}) \;=\; 1 \;+4z \;+36z^2 \;+400z^3 \;+4900z^4 \;+63504z^5 \;+853776z^6 \;+11778624z^7 \;+\cdots \tag{10}$$

is equal to the complete elliptic integral of the first kind

$$\mathsf{Diag}(\mathcal{F}) \;=\; \sum_{m \ge 0} \binom{2m}{m}^2 \cdot z^m \;=\; \frac{2}{\pi} \cdot \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - 16z\sin^2(\vartheta)}} \;=\; {}_2F_1\Bigl([\frac{1}{2},\, \frac{1}{2}],\, [1],\, 16\, z\Bigr), \tag{11}$$

which is a *transcendental* function.

Less obvious examples (see [65]) are

$$\mathsf{Diag}\Bigl(\frac{1}{1 - z_1 - z_2 - z_3 - z_1 z_2 - z_2 z_3 - z_3 z_1 - z_1 z_2 z_3}\Bigr) \;=\; \frac{1}{1-z} \cdot {}_2F_1\Bigl([\frac{1}{3},\, \frac{2}{3}],\, [1],\, \frac{54\, z}{(1-z)^3},\Bigr), \tag{12}$$

and

$$\mathsf{Diag}\Bigl(\frac{(1-z_1)(1-z_2)(1-z_3)}{1 \;-2\;(z_1+z_2+z_3)\;+3\;(z_1z_2+z_2z_3+z_3z_1)\;-4\,z_1z_2z_3}\Bigr) \;=\; 1 \;+6 \cdot \int_0^z {}_2F_1\Bigl([\frac{1}{3},\, \frac{2}{3}],\, [2],\, \frac{27\, w \cdot (2\;-3\, w)}{(1\;-4\, w)^3}\Bigr) \cdot \frac{dw}{(1\;-4w)\,(1\;-64w)}. \tag{13}$$

It was shown by Christol [66, 67, 68] that the diagonal $\mathsf{Diag}(\mathcal{F})$ of *any* rational function $\mathcal{F}$ is *D-finite*, in the sense that it satisfies a linear differential equation with polynomial coefficients¶. Moreover, the diagonal of any algebraic power series in $\mathbb{Q}[[z_1, \ldots, z_n]]$ is a G-function *coming from geometry*, i.e. it satisfies the Picard-Fuchs type differential equation associated with some one-parameter family of algebraic varieties. Diagonals of algebraic power series thus appear to be a *distinguished class* of G-functions♯. It will be seen below (see (3.5)) that algebraic functions with n variables can be seen as diagonals of rational functions with $2n$ variables. Thus diagonals of rational functions also appear to be a *distinguished class* of G-functions. It is worth noting that this distinguished class is stable by the Hadamard product: the *Hadamard product of two diagonals of rational functions is the diagonal of rational function.*

An immediate, but important property of diagonals of rational functions in $\mathbb{Q}[[z_1, \ldots, z_n]]$ is that they are *globally bounded*, which means that they have *integer coefficients* up to a simple change of variable $z \to N\, z$, where $N \in \mathbb{Z}$.

Furstenberg [69] showed that if K has positive characteristic, then the diagonal of any rational power series in $K[[z_1, \ldots, z_n]]$ is algebraic. Deligne [64, 58] extended this result to diagonals of algebraic functions. For instance, when $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1 z_2 - z_1 z_3)$, one gets modulo 3, modulo 5 and modulo 7 respectively

¶ A more general result was proved by Lipshitz [57]: *the diagonal of any D-finite series is D-finite*, see also [56].

♯ Such diagonals are solutions of G-operators. They are functions that are always algebraic mod. a prime p. They fill the gap between algebraic functions and G-series: they can be seen as *generalisations of algebraic functions.*

$$\mathsf{Diag}(\mathcal{F}) \quad \bmod 3 \;=\; 1 \;+ z \,+ z^3 \,+ z^4 \,+ z^9 \,+ z^{10} \,+ z^{12} \,+ z^{13} \;+\; \cdots$$
$$= \; \frac{1}{\sqrt{1\,+ z}} \quad \bmod 3,$$

$$\mathsf{Diag}(\mathcal{F}) \quad \bmod 5 \;=\; 1 \;+ 4\,z \;+ z^2 \,+ 4z^5 \,+ z^6 \,+ 4z^7 \,+ z^{10} \,+ 4z^{11} \,+ z^{12} \;+\; \cdots$$
$$= \; \frac{1}{\sqrt[4]{1\,-z\,+ z^2}} \quad \bmod 5,$$

$$\mathsf{Diag}(\mathcal{F}) \quad \bmod 7 \;=\; 1 \;+ 4\,z \;+ z^2 \,+ z^3 \,+ 4z^7 \,+ 2z^8 \,+ 4z^9 \;+\; \cdots$$
$$= \; \frac{1}{\sqrt[6]{1\,+ 4z\,+ z^2\,+ z^3}} \quad \bmod 7.$$

More generally, for any prime p, one has

$$\mathsf{Diag}(\mathcal{F}) \quad \bmod p \quad = \quad P(z)^{1/(1-p)} \quad \bmod p \qquad (14)$$

where the polynomial $P(z)$ is nothing but [70, 71, 72]

$$P(z) \;=\; {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], 16\,z\Big)^{1-p} \quad \bmod\, p$$
$$= \; \sum_{n=0}^{(p-1)/2} \binom{p\,-1/2}{n}^2 \cdot (16\,z)^n \quad \bmod\, p. \qquad (15)$$

For instance, modulo 11, the polynomial (15) reads:

$${}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], 16\,z\Big)^{-10} \quad \bmod 11 \;=\; 1 + 4\,z \,+ 3\,z^2 \,+ 4\,z^3 + 5\,z^4 \,+ z^5 \quad \bmod 11.$$

Interestingly enough, the polynomial modulo p

$$P\Big(\frac{\lambda}{16}\Big) \;=\; \sum_{n=0}^{(p-1)/2} \binom{p\,-1/2}{n}^2 \cdot \lambda^n, \qquad (16)$$

is [73, 74], up to a sign $(-1)^{(p-1)/2}$, the *Hasse invariant*† of $y^2 = x\cdot(1-x)\cdot(\lambda-x)$.

Note, however, that the Furstensberg-Deligne result [69, 64], that we illustrate, here, with $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1 z_2 - z_1 z_3)$, goes far beyond the case of hypergeometric functions for which simple closed formulae can be displayed.

3.3. Hadamard, and other products

Let us also recall the notion of *Hadamard product* [62, 75] of two series, that we will denote by a star.

$$\text{If} \qquad f(x) \,=\, \sum_{n=0}^{\infty} a_n \cdot x^n, \qquad g(x) \,=\, \sum_{n=0}^{\infty} b_n \cdot x^n, \qquad \text{then:}$$

$$f(x) \,\star\, g(x) \;=\; \sum_{n=0}^{\infty} a_n \cdot b_n \cdot x^n. \qquad (17)$$

† Note Igusa's sentence "Hence the elliptic differential of the first kind has only one period, and that is $A(\lambda)$, up to an arbitrary differential constant. This version of Hasse invariant has not yet been explicitly remarked."

The notion of diagonal of a function and the notion of Hadamard product are obviously related:

$$\mathsf{Diag}\Big(f_1(x_1) \cdot f_2(x_2) \, \cdots \, f_n(x_n)\Big) \quad = \quad f_1(x) \star f_2(x) \star \cdots \star f_n(x). \qquad (18)$$

In other words, the diagonal of a product of functions with separate variables is equal to the Hadamard product of these functions in a common variable. In particular, the Hadamard product of n rational (or algebraic, or even D-finite) power series is D-finite.

The Hadamard product of two series with integer coefficients is straightforwardly a series with integer coefficients. Furthermore, the *Hadamard product of two operators*, annihilating two series, defined as the (minimal order, monic) linear differential operator annihilating the Hadamard product of these two series, is a *product compatible with a large number of structures and concepts* that naturally occur in lattice statistical mechanics. For instance, the Hadamard product of two globally nilpotent [28] operators is *also globally nilpotent*.

Let us introduce another product, namely the *Hurwitz* (shuffle) *product*† of two series which is defined as [77, 78, 76]:

$$HurwitzProd\Big(\sum_n \alpha_n \cdot x^n, \, \sum_n \beta_n \cdot x^n\Big) \quad = \quad \sum_n \sum_m \binom{n+m}{n} \cdot \alpha_n \, \beta_m \cdot x^{n+m}.$$

A very simple example is, for instance,

$$HurwitzProd\Big(\frac{1}{1 - a \cdot x}, \, \frac{1}{1 - b \cdot x}\Big) \quad = \quad \frac{1}{1 - (a+b) \cdot x}. \qquad (19)$$

Again, we have a remarkable compatibility property between the diagonal and the Hurwitz product. The Hurwitz product of two series that are diagonals of two power series $A(x_1, x_2, \, \ldots, x_n)$ and $B(y_1, y_2, \, \ldots, y_m)$ can itself be seen as the diagonal of a power series [68], that is very close to the product of these two power series‡:

$$HurwitzProd\Big(\mathsf{Diag}(A(x_1, x_2, \, \ldots, x_n)), \, \mathsf{Diag}(B(y_1, y_2, \, \ldots, y_m))\Big) \qquad (20)$$

$$= \quad \mathsf{Diag}\Big(\frac{A(x_1, x_2, \, \ldots, x_n) \cdot B(y_1, y_2, \, \ldots, y_m)}{1 \, - \, t \cdot x_1 \, x_2 \, \cdots x_n \, y_1 \, y_2 \, \cdots y_m}\Big).$$

where t is an additional variable. In fact there exists an infinite number of products that enjoy a similar property of compatibility with the diagonal. The most general products of series compatible with the diagonal are, beyond the Hadamard and Hurwitz products:

$$GeneralProd\Big(\sum_n \alpha_n \cdot x^n, \sum_n \beta_n \cdot x^n\Big) \quad = \quad \sum_n \sum_m p(n, m) \cdot \alpha_n \, \beta_m \cdot x^{n+m},$$

where the $p(n, m)$'s are coefficients of *any rational function* of two variables $R(x, y)$:

$$R(x, y) \quad = \quad \sum_n \sum_m p(n, m) \cdot x^n \, y^m. \qquad (21)$$

Special cases are *Lamperti's product* [79, 76] and *Trjitzinsky's product* [80].

† The Hurwitz product of two algebraic functions is not algebraic in general, e.g. the Hurwitz square of $(1-4z)^{-1/2}$ is equal to ${}_2F_1([1/2, 1/2], [1], 16z\,(1-4z))$. However, the Hurwitz product of two algebraic functions is actually an algebraic function *modulo a prime p*, cf. Prop. 8 in [76]. The Hurwitz product of a rational and of an algebraic power series with coefficients in $\mathbb{Q}$ is algebraic, cf. Prop. 3 & 7 of [76].

‡ Note a misprint in the second equation of page 68 of [68].

3.4. Chiral Potts examples

Let us consider, for instance, the Hadamard cube of a simple algebraic function

$$
\begin{aligned}
{}_3F_2\left(\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right], [1, 1]; z\right) \; &= \; \mathsf{Diag}\Big((1 - x)^{-1/3} \star (1 - y)^{-1/3} \star (1 - z)^{-1/3}\Big) \\
&= \; {}_1F_0([\frac{1}{3}], []; z) \star {}_1F_0([\frac{1}{3}], []; z) \star {}_1F_0([\frac{1}{3}], []; z) \\
&= \; (1 - z)^{-1/3} \star (1 - z)^{-1/3} \star (1 - z)^{-1/3}. \qquad (22)
\end{aligned}
$$

It is globally bounded:

$$
\begin{aligned}
{}_3F_2\left(\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right], [1, 1]; 3^5 x\right) \; &= \; 1 + 9x + 648x^2 + 74088x^3 + 10418625x^4 \\
&+ 1648059777x^5 + 281268868608x^6 + 50621016116736x^7 + \cdots
\end{aligned}
$$

Other examples, related to the chiral Potts model and its associated Fermat curves [59] are ${}_2F_1([1/2, N/3], [1], 36t)$ and ${}_2F_1([1/2, N/5], [1], 100t)$ which have series expansions *with integer coefficients*, or, more generally:

$$
{}_3F_2\left(\left[\frac{t}{N}, \frac{q}{N}, \frac{s}{N}\right], [1, 1]; x\right) \; = \; (1 - x)^{-t/N} \star (1 - x)^{-q/N} \star (1 - x)^{-s/N}.
$$

3.5. Furstenberg's result on algebraic functions

It was shown by Furstenberg [69] that *any algebraic series* in one variable can be written as the *diagonal of a rational function of two variables* (however, this representation is, by no means unique). For instance,

$$
f \; = \; \frac{x}{\sqrt{1-x}} \; = \; x + \frac{1}{2}x^2 + \frac{3}{8}x^3 + \frac{5}{16}x^4 + \frac{35}{128}x^5 + \frac{63}{256}x^6 + \cdots \qquad (23)
$$

is the diagonal of $(2xy - cx + cy)/(x + y + 2)$ for *any rational number c*.

The basis of Furstenberg's result is the fact that if $f(x)$ is a power series without constant term, and is a root of a polynomial $P(x, y)$ such that $P_y(0, 0) \neq 0$, then

$$
f(x) \; = \; \mathsf{Diag}\left(y^2 \cdot \frac{P_y(xy, y)}{P(xy, y)}\right) \qquad \text{where:} \qquad P_y \; = \; \frac{\partial P}{\partial y}. \qquad (24)
$$

When $P_y(0, 0) = 0$, this formula is not true anymore. For instance, it does not apply to the algebraic function $f = x/\sqrt{1-x}$, annihilated by $P = (x - 1)y^2 + x^2$, since the diagonal of $y^2 P_y(xy, y)/P(xy, y) = 2y(xy - 1)/(x^2 + xy - 1)$ is zero. However, Furstenberg's result still holds. A way of seeing this on our example is to observe that $g = f - x - \frac{1}{2}x^2$ is an algebraic series annihilated by a polynomial Q such that $Q(0, 0) = 0$ and $Q_y(0, 0) \neq 0$. This reasoning, which extends to the general situation, yields the following rational function whose diagonal equals f:

$$
\begin{aligned}
&x\,y \cdot \frac{\mathcal{P}(x, y)}{\mathcal{Q}(x, y)} \qquad \text{where:} \qquad (25) \\
&\mathcal{P}(x, y) \; = \; 16x^3y^5 + 4 \cdot (3x - 4) \cdot x^2y^4 + 4 \cdot (3 + x) \cdot x^2y^3 \\
&\qquad + (12x - 24 + x^2)\, xy^2 + 5yx^2 + 6x - 16, \\
&\mathcal{Q}(x, y) \; = \; 8x^2y^3 + 8 \cdot (x - 1) \cdot xy^2 + 2 \cdot (x + 4) \cdot xy + 6x - 16.
\end{aligned}
$$

When compared to $(2xy - cx + cy)/(x + y + 2)$, whose diagonal is also f, this shows that Furstenberg's proof does not necessarily produce the easiest rational function.

Furstenberg's result has been generalised to algebraic power series in an *arbitrary number of variables*: any power series ¶ in $\mathbb{Q}[[x_1, ..., x_n]]$ algebraic over $\mathbb{Q}(x_1, ..., x_n)$ is the diagonal of a rational function with $2n$ variables (see Denef and Lipshitz [81]).

4. Selected n-fold integrals are diagonals of rational functions

Among many multiple integrals that are important in various domains of mathematical physics, let us consider the n-particle contribution to the magnetic susceptibility of the Ising model which we denote $\tilde{\chi}^{(n)}(w)$. They are given by $(n-1)$-dimensional integrals [14, 82]:

$$\tilde{\chi}^{(n)}(w) \quad = \quad \frac{(2w)^n}{n!} \Big(\prod_{j=1}^{n-1} \int_0^{2\pi} \frac{d\Phi_j}{2\pi} \Big) \cdot Y \cdot \frac{1+X}{1-X} \cdot X^{n-1} \cdot G \, , \tag{26}$$

where, defining Φ_0 by $\sum_{i=0}^{n-1} \Phi_i = 0$, we set

$$X = \prod_{i=0}^{n-1} x_i, \qquad x_i = \frac{2w}{A_i + \sqrt{A_i^2 - 4w^2}}, \qquad Y = \prod_{i=0}^{n-1} y_i, \qquad y_i = \frac{1}{\sqrt{A_i^2 - 4w^2}},$$

$$G = \prod_{0 \le i < j \le n-1} \frac{2 - 2\cos(\Phi_i - \Phi_j)}{(1 - x_i x_j)^2}, \qquad \text{where:} \qquad A_i = 1 - 2w\cos(\Phi_i). \tag{27}$$

The integrality property (1) had been checked [15] for the first $\tilde{\chi}^{(n)}$ and inferred [28] for generic n. We are going to *prove it*‡ *for any integer n, showing a much fundamental result, namely that all the $(n-1)$-fold integrals $\tilde{\chi}^{(n)}$'s are very special*: they are actually *diagonals of rational functions*.

4.1. $\tilde{\chi}^{(3)}$'s as a toy example

At first sight the $\tilde{\chi}^{(n)}$'s are involved transcendental holonomic functions. Could it be possible that they correspond to the *distinguished class* [58] of G-functions, generalising algebraic functions, which have an interpretation as diagonals of multivariate algebraic functions (and consequently diagonals of rational functions with twice more variables)? If this is the case, then the series of the $\tilde{\chi}^{(n)}$'s must *necessarily reduce modulo any prime to an algebraic function* (see (**??**)). The $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(2)}$ contributions being too degenerate (a rational function and a too simple elliptic function), let us consider the first non-trivial case, namely $\tilde{\chi}^{(3)}$. Its series expansion has already been displayed in [14]. It reads $\tilde{\chi}^{(3)}/8 \; = \; w^9 \cdot F(w)$ with:

$$F(w) \;\; = \;\; 1 \, + \, 36\, w^2 \, + \, 4\, w^3 \, + \, 884\, w^{13} \, + \, 196\, w^5 \, + \, 18532\, w^6 \, + \, 6084\, w^7 \;\; + \; \cdots$$

Since we have obtained the exact ODE satisfied by $\tilde{\chi}^{(3)}$ we can produce as many coefficients as we want in its series expansion. Let us consider this series modulo the prime $p = 2$. It now reads the lacunary series

$$F(w) \quad \bmod 2 \;\; = \;\; 1 \, + w^8 + w^{24} + w^{56} + w^{120} + w^{248} + w^{504} \, + w^{1016} \; + \cdots \, ,$$

¶ In the one-variable case, Puiseux series could be considered but only after ramifying the variable.
‡ Actually we only prove global boundedness for the Taylor expansion $\tilde{\chi}^{(n)}(w) \; = \; \sum a_k \, w^k$. However, looking et the process more carefully, and, in particular, adding corresponding properties on the set $\mathcal{T}_n$ below, one can find out that only powers of 2 appear in the denominators of the a_k: the rescaling factor ("Eisenstein constant") is 2 or 4 according to the fact that one considers high or low temperature series [15, 27], and that $\sum a_k w^k$ do converge for $|w| < 1/4$.

solution of the functional equations on $F(w)$ or, with $z = w^8$, on $G(z) = 1 + w^8 \cdot F(w)$

$$F(w) \,=\, 1 \,+\, w^8 \cdot F(w^2), \qquad\qquad G(z) \,=\, z \,+\, G(z^2), \tag{28}$$

where one recognises, with equation $G(z) = z + G(z^2)$, Furstenberg's example [69] of the simplest algebraic function in characteristic 2§. In fact $H(w) = w^9 F(w)$ is solution of the quadratic equation:

$$H(w)^2 \,+ w \cdot H(w) \,+ w^{10} \quad = \quad 0 \qquad \text{mod } 2. \tag{29}$$

The calculations are more involved modulo $p = 3$. Indeed, $H(w) = \tilde{\chi}^{(3)}(w)/8$ satisfies, modulo 3, the polynomial equation of degree nine

$$p_9 \cdot H(w)^9 \,+\, w^6 \cdot p_3 \cdot H(w)^3 \,+\, w^{10} \cdot p_1 \cdot H(w) \\ +\, w^{19} \cdot p_0^{(1)} \cdot p_0^{(2)} \quad = \quad 0, \tag{30}$$

where:

$$p_9 \,=\, (w+1)^3\,(w^2+1)^{18}\,(w-1)^{24},$$
$$p_3 \,=\, (w^2+1)^{18}\,(1-w)^{15}\,(w^4-w^2-1), \qquad p_1 \,=\, (w^2+1)^{20}\,(1-w)^{13},$$
$$p_0^{(1)} \,=\, w^6+w^5+w^4-w^2-w+1, \tag{31}$$
$$p_0^{(2)} \,=\, w^{37}-w^{36}+w^{35}-w^{33}+w^{31}-w^{30}+w^{28}+w^{27}+w^{24}-w^{23}+w^{22} \\ -w^{21}-w^{18}-w^{16}+w^{14}-w^{12}-w^{11}-w^{10}+w^7-w^5-w^3-1.$$

The calculations are even more involved modulo larger primes. The series for $\tilde{\chi}^{(3)}$ mod. 5 reads:

$$\tilde{\chi}^{(3)} \quad = \quad w^9 \,+ w^{11} \,+ 4\,w^{12} \,+ 4\,w^{13} \,+ w^{14} \,+ 2\,w^{15} \,+ 4\,w^{16} \,+\, \cdots \tag{32}$$

The (minimal order) linear differential operator annihilating the $\tilde{\chi}^{(3)}$ series mod. 5, reads†:

$$(x+1)\,(x^2+x+1)\,(x+2)\cdot x^4 \cdot D_x^4 \,+ 2\,x^3 \cdot (x^3+2\,x^2+4\,x+4)\,(x+4)\cdot D_x^3 \\ + x^2 \cdot (x^4+3\,x^3+4)\cdot D_x^2 \,+ 4 \cdot (x^4+3)\cdot x \cdot D_x \,+ 3 \tag{33}$$

If one can easily get this linear differential operator, finding the minimal polynomial of $\tilde{\chi}^{(3)}$ modulo 5, generalising (29) or (30), such that $P(\tilde{\chi}^{(3)}(w), w) = 0$ mod. 5, requires a *very large* number of coefficients. Since the series (32) starts with w^9, it is more convenient to consider the polynomial $\tilde{P}(\kappa, w)$, relating $\kappa = \tilde{\chi}^{(3)}(w)/w^9$ and w. This (minimal) polynomial‡ is a polynomial of degree 50 in κ, and degree 832 in w, sum of 4058 monomials. This (minimal) polynomial of the form:

$$\tilde{P}(\kappa, w) \;=\; P_{50}^{(832)}(w)\cdot\kappa^{50} \,+\, P_{30}^{(652)}(w)\cdot\kappa^{30} \,+\, P_{26}^{(612)}(w)\cdot\kappa^{26} \\ +\, P_{25}^{(601)}(w)\cdot\kappa^{25} \,+\, P_{10}^{(472)}(w)\cdot\kappa^{10} \,+\, P_{6}^{(432)}(w)\cdot\kappa^{6} \,+\, P_{5}^{(421)}(w)\cdot\kappa^{5} \\ +\, P_{2}^{(392)}(w)\cdot\kappa^{2} \,+\, P_{1}^{(381)}(w)\cdot\kappa \,+\, P_{0}^{(369)}(w), \tag{34}$$

where the $P_n^{(m)}(w)$'s are polynomials of degree m in w, and where the head polynomial reads:

$$P_{50}^{(832)}(w) \,=\, w^{382}\cdot(w+2)^{20}\,(w^2+2\,w+4)^{75}\,(w+1)^{70}\,(w+4)^{20} \\ \times\,(w^2+3\,w+4)^{75}\,(w^4+4\,w^3+w+1)^{10}. \tag{35}$$

§ Modulo the prime $p = 2$, the previous functional equation becomes $G(z) = z + G(z)^2$.
† This operator is of zero 5-curvature [28].
‡ This polynomial has been checked with a series (32) of 380000 coefficients.

This (minimal) polynomial is a factor of a much larger polynomial (in 1, κ, κ^5, κ^{5^2}, and κ^{5^3}) of a more "p-adic nature", which is of degree 125 in κ, sum of 3559 monomials, and of degree 1941 in w.

One can imagine, in a first step that the $\tilde{\chi}^{(3)}$ series mod. *any prime* p are *also algebraic functions*, and, in a second step, that $\tilde{\chi}^{(3)}$ may be the diagonal of a rational function. In fact we are going to show, in the next section, a stronger result: the $\tilde{\chi}^{(n)}$'s are *actually diagonals of rational functions, for any integer* n.

4.2. The $\tilde{\chi}^{(n)}$'s are diagonals of rational functions

Let us, now, consider the general case where n is an arbitrary integer.

With the change of variable $z_i = \exp(\iota\Phi_i)$ (where $\iota^2 = -1$), one clearly gets

$$\prod_{i=0}^{n-1} z_i \,\, = \,\, 1 \, , \qquad \qquad {{dz_j} \over {z_j}} \,\, = \,\, \iota \, d\Phi_j \, , \qquad (36)$$

$$2\cos(\Phi_i) \,\, = \,\, z_i \, + {{1} \over {z_i}} \, , \qquad \qquad 2\cos(\Phi_i - \Phi_j) \,\, = \,\, {{z_i} \over {z_j}} + {{z_j} \over {z_i}} \, ,$$

and (26) becomes

$$\tilde{\chi}^{(n)}(w) \,\, = \,\, {{(2w)^n} \over {n!}} \, \Bigl(\prod_{j=1}^{n-1} {{1} \over {2\iota\pi}} \oint_{\mathcal{C}} {{dz_j} \over {z_j}}\Bigr) \cdot F(w, z_1, \ldots, z_{n-1}) \, , \qquad (37)$$

where $\mathcal{C}$ is the path "turning once counterclockwise around the unit circle" and where F is algebraic over $\mathbb{Q}(w, z_1, \, \ldots, \, z_{n-1})$ and reads:

$$F(w, \, z_1, \ldots, \, z_{n-1}) \,\, = \,\, Y \cdot X^{n-1} \cdot {{1 \, + \, X} \over {1 \, - \, X}} \cdot G. \qquad (38)$$

Now, let us suppose that F *is analytic*† *at the origin*, namely that it has a Taylor expansion (5). Then applying $(n-1)$ times the residue formula, one finds

$$\tilde{\chi}^{(n)}(w) \,\, = \,\, \mathsf{Diag}\Bigl({{(2\, z_1 \cdots z_n)^n} \over {n!}} \, \cdot F(z_1 \cdots z_{n-1} z_n, \,\, z_1, \, \ldots, \, z_{n-1})\Bigr). \qquad (39)$$

To check that this is actually true, we introduce an auxiliary set, namely $\mathcal{T}_n$ the subset of $\mathbb{Q}[z_1, \, \ldots, \, z_{n-1}, z_1^{-1}, \ldots, z_n^{-1}][[w]]$, consisting of series

$$f(w, \, z_1, \, \ldots, \, z_{n-1}) \,\, = \,\, \sum_{m=0}^{\infty} P_m \cdot w^m,$$

where P_m belongs to $\mathbb{Q}[z_1, \ldots, z_{n-1}, z_1^{-1}, \ldots, z_n^{-1}]$ and is such that $\widetilde{f}(z_1, \cdots, z_n)$ belongs to $\mathbb{Q}[z_1, ..., z_{n-1}][[z_n]] \,\subset\, \mathbb{Q}[[z_1, ..., z_n]]$, where $\widetilde{f}$ is defined by:

$$\widetilde{f}(z_1, \cdots, z_n) \,\, = \,\, f(z_1 \cdots z_n, \,\, z_1, \cdots, z_{n-1}) \qquad (40)$$

In other words, we ask the degree of P_m, in each of the z_i^{-1}, to be at most m. Then to prove that $\widetilde{F}$ has a Taylor expansion, we only have to verify that F belongs

† One could consider Laurent, instead of Taylor, expansions, but this is a slight generalisation. The rational function $1/(x+xy)$ would become allowed but rational functions like $1/(x+y)$ would remain forbidden. For similar purposes, B. Adamczewski and J. P. Bell [58] recently used a generalised notion of Laurent expansions in the several variables case due to Sathaye [83], see also [84] for various other generalisations.

to $\mathcal{T}_n$. Checking this is a straightforward step-by-step computation on auxiliary functions:

$$A_i \,=\, 1 \,- w \cdot (z_i + \frac{1}{z_i}), \qquad \text{for:} \quad 1 \le i \le n-1,$$
$$A_i \,=\, 1 \,- w \cdot \Bigl(\frac{1}{z_1 \cdots z_{n-1}} + z_1 \cdots z_{n-1}\Bigr), \qquad \text{for:} \quad i \,=\, 0, \qquad (41)$$
$$\tilde{A}_i \,=\, 1 \,- z_1 \,\cdots\, z_{i-1} \cdot z_{i+1}\, z_n \cdot (z_i^2 + 1), \qquad \text{for:} \quad 1 \le i \le n-1,$$
$$\tilde{A}_i \,=\, 1 \,- z_n \cdot \bigl(1 + z_1^2 \cdots z_{n-1}^2\bigr), \qquad \text{for:} \quad i \,=\, 0.$$

Hence $A_i \in \mathcal{T}_n$. The set $\mathcal{T}_n$ being clearly a $\mathbb{Q}[w]$-algebra, $A_i^2 - 4w^2 \in \mathcal{T}_n$. But $\mathcal{T}_n$ is complete for the w-adic valuation. In particular, it is stable by the operations‡

$$f(w) \quad \longrightarrow \quad \frac{1}{1 + w \cdot f(w)} \,=\, 1 \,- w \cdot f(w) \,+ \,\cdots, \qquad \text{and:} \qquad (42)$$
$$f(w) \quad \longrightarrow \quad \sqrt{1 + w \cdot f(w)} \,=\, 1 \,+ \frac{1}{2}\, w \cdot f(w) \,+ \,\cdots \qquad (43)$$

So to be sure that the inverse or the square root of some function in $\mathcal{T}_n$ is also in $\mathcal{T}_n$ we have only to check that its first Taylor coefficient is actually 1:

$$A_i^2 - 4\, w^2 \,=\, 1 \,- 2\, w \cdot (z_i + \frac{1}{z_i}) \,+ w^2 \cdot (z_i - \frac{1}{z_i})^2, \qquad (44)$$
$$\text{hence} \qquad \sqrt{A_i^2 - 4w^2} \,=\, 1 \,+\, \cdots \,\in\, \mathcal{T}_n,$$

$$y_i \,=\, \frac{1}{\sqrt{A_i^2 - 4w^2}} \,=\, 1 \,+ \cdots \,\in\, \mathcal{T}_n, \qquad Y \,=\, 1 \,+ \cdots \,\in\, \mathcal{T}_n,$$
$$x_i \,=\, \frac{2\, w}{A_i + \sqrt{A_i^2 - 4w^2}} \,=\, w \,+ \cdots \,\in\, \mathcal{T}_n, \qquad x_i\, x_j \,=\, w^2 \,+ \cdots \,\in\, \mathcal{T}_n,$$
$$X \,=\, w^n \,+ \cdots \,\in\, \mathcal{T}_n, \qquad \frac{1+X}{1-X} \,=\, 1 \,+ \cdots \,\in\, \mathcal{T}_n,$$
$$G \,=\, \prod_{0 \le i < j \le n-1} \frac{(z_i - z_j)^2}{(1 - x_i x_j)^2 \cdot z_i\, z_j} \,=\, \prod_{0 \le i < j \le n-1} \frac{(z_i - z_j)^2}{z_i\, z_j} \,+ \cdots \,\in\, \mathcal{T}_n.$$

From the definition of Φ_0 which implies $\prod_{i=0}^{n-1} z_i \,=\, 1$ we also have

$$\prod_{0 \le i < j \le n-1} z_i z_j \,=\, \Bigl(\prod_{i=0}^{n-1} z_i\Bigr)^{n-1} \,=\, 1, \qquad (45)$$

enabling to rewrite G in other ways.

Thus, F belongs to $\mathcal{T}_n$ and it makes sense to take its diagonal. As F is algebraic, $\tilde{\chi}^{(n)}$ *is the diagonal of an algebraic function* of n variables and, consequently, *the diagonal of a rational function* of $2\,n$ variables.

We thus see that we can actually *find explicitly* the algebraic function such that its diagonal is the n-fold integrals $\tilde{\chi}^{(n)}$: it is *nothing but the integrand* of the n-fold integral, up to trivial transformations, namely (38).

Remark : $\tilde{\chi}^{(n)}$ is a solution of a linear differential equation, and has a radius of convergence equal to $1/4$ in w. Among the other solutions of this equation, there

‡ More generally it is stable by the operations $f \,\rightarrow\, (1 \,+\, w\, f)^{\delta}$ for $\delta \,\in\, \mathbb{Q}$.

is the function obtained by changing the radical appearing in x_i into its opposite. A priori there are 2^n ways to do this, hence 2^n new solutions but, not all distinct. At first sight, for these new solutions, the x_i's are no longer in $\mathcal{T}_n$.

In fact, we find some quite interesting structure. Let us consider, for instance, the case of $\tilde{\chi}^{(3)}$. If one considers other choices of sign in front of the nested square roots in the integrand, the series expansions of the corresponding n-fold integrals read:

$$\begin{aligned}
&w + 6\,w^2 + 28\,w^3 + 124\,w^4 + 536\,w^5 + 2280\,w^6 + 9604\,w^7 + 40164\,w^8 \\
&\qquad +167066\,w^9 + 692060\,w^{10} + 2857148\,w^{11} \,\, + \cdots \\
&w^2 + 6\,w^3 + 30\,w^4 + 140\,w^5 + 628\,w^6 + 2754\,w^7 + 11890\,w^8 + 50765\,w^9 \\
&\qquad +214958\,w^{10} + 904286\,w^{11} \,\, + \cdots
\end{aligned} \tag{46}$$

These two series expansions (46) are solutions of the same order-seven operator [14] L_7 as $\tilde{\chi}^{(3)}$.

We know that other forms (equivalent for integration purposes) of G exist (see [85, 86, 87]). For these forms, other choices of sign in front of the nested square roots give

$$\begin{aligned}
S(+,-,+) \,\,&=\,\, w + 6\,w^2 + 28\,w^3 + 126\,w^4 + 552\,w^5 + 2388\,w^6 + 10192\,w^7 \\
&\quad + 43238\,w^8 + 181936\,w^9 + 762836\,w^{10} + 3180800\,w^{11} \,\, + \cdots \\
S(+,+,-) \,\,&=\,\, w - 2\,w^2 - 20\,w^3 - 110\,w^4 - 552\,w^5 - 2536\,w^6 - 11428\,w^7 \\
&\quad - 49898\,w^8 - 216016\,w^9 - 920776\,w^{10} - 3905764\,w^{11} \,\, + \cdots \\
S(-,-,+) \,\,&=\,\, w + 6\,w^2 + 8\,w^3 + 14\,w^4 - 84\,w^5 - 596\,w^6 - 4004\,w^7 - 19610\,w^8 \\
&\quad - 99148\,w^9 - 447332\,w^{10} - 2068492\,w^{11} \,\, + \cdots \\
S(-,+,-) \,\,&=\,\, w + 10\,w^2 + 44\,w^3 + 202\,w^4 + 848\,w^5 + 3672\,w^6 + 15200\,w^7 \\
&\quad + 64310\,w^8 + 264424\,w^9 + 1104872\,w^{10} + 4523656\,w^{11} \,\, + \cdots
\end{aligned}$$

$$\begin{aligned}
S(+,-,-) \,\,&=\,\, w^2 + 3\,w^3 + 13\,w^4 + 47\,w^5 + 189\,w^6 + 707\,w^7 + 2800\,w^8 \\
&\quad + 10637\,w^9 + 41865\,w^{10} + 160535\,w^{11} \,\, + \cdots \\
S(-,+,+) \,\,&=\,\, w^4 + 4\,w^5 + 25\,w^6 + 103\,w^7 + 496\,w^8 + 2042\,w^9 + 9013\,w^{10} \\
&\quad + 36931\,w^{11} \,\, + \cdots
\end{aligned}$$

to be compared with $\tilde{\chi}^{(3)} \,=\, S(+,+,+) \,=\, S(-,-,-)$. These alternative series are not solutions of the order-seven operator [14] L_7 annihilating $\tilde{\chi}^{(3)}$, however, some linear combination of these series are solutions of L_7. For instance the linear combination $S(+,+,-) - S(-,+,+) - S(+,-,+)$, which is actually equal to $S(-,+,-) + 4\,S(+,-,-) - S(-,-,+)$, is solution of L_7.

Another form of G (again equivalent for integration purposes) gives:

$$\begin{aligned}
\tilde{S}(+,-,+) \,\,&=\,\, w + 8\,w^2 + 36\,w^3 + 164\,w^4 + 704\,w^5 + 3041\,w^6 + 12786\,w^7 \\
&\quad + 54067\,w^8 + 224864\,w^9 + 939709\,w^{10} + 3881708\,w^{11} \,\, + \cdots \\
\tilde{S}(+,+,-) \,\,&=\,\, w + 2\,w^2 - 6\,w^3 - 48\,w^4 - 314\,w^5 - 1555\,w^6 - 7626\,w^7 \\
&\quad - 34461\,w^8 - 155898\,w^9 - 678199\,w^{10} - 2957648\,w^{11} \,\, + \cdots \\
\tilde{S}(-,+,+) \,\,&=\,\, w^2 + 3\,w^3 + 14\,w^4 + 51\,w^5 + 214\,w^6 + 810\,w^7 + 3296\,w^8 \\
&\quad + 12679\,w^9 + 50878\,w^{10} + 197466\,w^{11} \, + \cdots
\end{aligned}$$

Again, these other alternative series are not solutions of L_7, but the linear combination $2\,\tilde{S}(-,+,+) + \tilde{S}(+,-,+) - \tilde{S}(+,+,-)$ is solution of L_7.

All these alternative series are, in fact, solution of a higher order linear differential operator that L_7 rightdivides.

One does remark that *all these alternative series* are, as $\tilde{\chi}^{(3)}$, series with *integer coefficients*.

4.3. More n-fold integrals of the Ising class and a simple integral of the Ising class

It is clear that the demonstration we have performed on the $\chi^{(n)}$'s can also be performed straightforwardly, mutatis mutandis, with other n-fold integrals of the "Ising class" like the n-fold integrals Φ_H in [82], which amounts to getting rid of the fermionic term G (see (41)), the $\chi_d^{(n)}$'s corresponding to n-fold integrals associated with the diagonal† susceptibility [18, 33] (the magnetic field is located on a diagonal of the square lattice), the $\Phi_D^{(n)}$'s in [17] which are simple integrals, and also for all the lattice Green functions displayed in [48, 52], and the list is far from being exhaustive. For instance, the simple integral $\Phi_D^{(n)}$ is the diagonal of the algebraic function:

$$\frac{2}{n!} \cdot (1-t^2)^{-1/2} \cdot \frac{G_n\, F_n^{n-1}}{G_n\, F_n^{n-1} \,-\, (2\,w\,t)^n} \quad - \,\frac{1}{n!}\,, \qquad \hbox{where:} \tag{47}$$

$$F_n \, = \, 1 \, -2\,w \, + (1-4\,w\,+4\,w^2\,-4\,w^2\,t^2)^{1/2}, \tag{48}$$

$$G_n \, = \, 1 \, -2\,w\,t \cdot T_{n-1}\Big(\frac{1}{t}\Big) \, + \Big(\Big(1 \, -2\,w\,t \cdot T_{n-1}\Big(\frac{1}{t}\Big)\Big)^2 \, -4\,w^2 \cdot t^2\Big)^{1/2},$$

and where $T_{n-1}(t)$ is the $(n-1)$-th Chebyshev polynomial of the first kind. The way we have obtained these Chebyshev results (47) is displayed in Appendix C.

The integral $\Phi_D^{(n)}(w)$ is the diagonal of an algebraic function of *two* variables, and also the diagonal of a rational function of *four* variables, and this *independently of the actual value of n*.

4.4. More general n-fold integrals as diagonals

More generally the demonstration we have performed on the $\tilde{\chi}^{(n)}$'s can be performed for *any* n-fold integral that can be recast in the following form:

$$\int_C \int_C \cdots \int_C \frac{dz_1}{z_1}\,\frac{dz_2}{z_2} \,\,\cdots\,\, \frac{dz_n}{z_n} \cdot \mathcal{A}\Big(x,\, z_1,\, z_2,\, \cdots,\, z_n\Big), \tag{49}$$

where the subscript C denotes the unit circle, and where $\mathcal{A}$ denotes an algebraic function of the n variables, which (this is the crucial ingredient), as a function of several variables x and the z_i's, has an *analytical* expansion at $(x,\, z_1,\, z_2,\, \cdots z_n) = (0,\, 0,\, 0,\, \cdots,\, 0)$:

$$\mathcal{A}\Big(x,\, z_1,\, z_2,\, \cdots,\, z_n\Big) \quad = \tag{50}$$
$$\sum_{m\,=0}^{\infty}\,\sum_{m_1\,=0}^{\infty}\,\sum_{m_2\,=0}^{\infty}\,\cdots\,\sum_{m_n\,=0}^{\infty} A_{m,\,m_1,\,m_2,\,\cdots,\,m_n} \cdot z_1^{m_1}\, z_2^{m_2} \,\,\cdots\,\, z_n^{m_n} \cdot x^m.$$

† Of course this "diagonal wording" should not be confused with the notion of diagonal of a function.

Consequently, an extremely large set of n-fold integrals occurring in theoretical physics (lattice statistical mechanics, enumerative combinatorics, number theory, differential geometry, ...) can actually be seen to be *diagonal of rational functions*. Consequently these n-fold integrals correspond to series expansions (in the variation parameter x) that are *globally bounded* (can be written after one rescaling into series with integer coefficients), and are solutions of *globally nilpotent* [28] linear differential operators.

Such a general n-fold integral is, thus, the diagonal of an algebraic function (or of a rational function with twice more variables [81]) which is essentially the integrand of such n-fold integral. Furthermore, such a general n-fold integral is solution of a (globally nilpotent) linear differential operator, that can be obtained exactly from the integrand, using the creative telescoping method (see Appendix D).

Finally, in the case of Calabi-Yau ODEs (see below), these functions can be interpreted as periods of Calabi-Yau varieties, these algebraic varieties being essentially the integrand of such n-fold integrals. The integrand is thus the key ingredient to wrap, in the same bag, algebraic geometry viewpoint, differential geometry viewpoint and analytic and arithmetic approaches (series with integer coefficients).

5. Calabi-Yau ODEs

Calabi-Yau ODEs have been defined in [88] as order-four linear differential ODEs that satisfy the following conditions: they are maximal unipotent monodromy [89, 90] (MUM), they satisfy a "Calabi-Yau condition" which amounts to imposing that the exterior squares of these order-four operators are of order *five* (instead of the order six one expects in the generic case), the series solution, analytic at $x = 0$, is globally bounded (can be reduced to integer coefficients), the series of their nome and Yukawa coupling are globally bounded‡. In the literature, one finds also a cyclotomic condition on the monodromy at the point at ∞, $x = \infty$, and/or the conifold† character of one of the singularities [92].

Let us recall that a linear ODE has MUM (maximal unipotent monodromy [32, 91]) if all the exponents at (for instance) $x = 0$ are zero.

In a hypergeometric framework the MUM condition amounts to restricting to hypergeometric functions of the type ${}_{n+1}F_n([a_1, a_2, \cdots a_n], [1, 1, \cdots 1]; x)$, since the indicial exponents at $x = 0$ are the solutions of $\rho\,(\rho + b_1 - 1)\cdots(\rho + b_n - 1) = \rho^{n+1} = 0$, where the b_j are the lower parameters which are here all equal to 1.

Let us consider a MUM order-four linear differential operator. The four solutions y_0, y_1, y_2, y_3 of this order-four linear differential operator read:

$$y_0, \qquad y_1 = y_0 \cdot \ln(x) + \tilde{y}_1, \qquad y_2 = y_0 \cdot \frac{\ln(x)^2}{2} + \tilde{y}_1 \cdot \ln(x) + \tilde{y}_2,$$

$$y_3 = y_0 \cdot \frac{\ln(x)^3}{6} + \tilde{y}_1 \cdot \frac{\ln(x)^2}{2} + \tilde{y}_2 \cdot \ln(x) + \tilde{y}_3,$$

where y_0, $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{y}_3$ are analytical at $x = 0$ (with also $\tilde{y}_1(0) = \tilde{y}_2(0) = \tilde{y}_3(0) = 0$).

‡ The instantons numbers are integers.
† The local exponents are $0, 1, 1, 2$. For the cyclotomic condition on the monodromy at ∞ see Proposition 3 in [91].

The nome of this linear differential operator reads:

$$q(x) \;=\; \exp\Bigl(\frac{y_1}{y_0}\Bigr) \;=\; x \cdot \exp\Bigl(\frac{\tilde{y}_1}{y_0}\Bigr). \qquad (51)$$

Calabi-Yau ODEs have been defined as being MUM, thus having one solution analytical at $x = 0$. As far as Calabi-Yau ODEs are concerned, the fact that this solution analytical at $x = 0$ has an integral representation, and, furthermore, an integral representation of the form (49) together with (50), is far from clear, even if one may have a "Geometry-prejudice" that this solution, analytical at $x = 0$, can be interpreted as a "Period" and "Derived From Geometry" [29, 30, 93].

Large tables of Calabi-Yau ODEs have been obtained by Almkvist et al. [91, 94, 95]. It is worth noting that the coefficients A_n of the series corresponding to the solution analytical at $x = 0$, are, most of the time, *nested sums of product of binomials*, less frequently nested sums of product of binomials and of harmonic numbers¶ H_n, and, in rare cases, no "closed formula" is known for these coefficients.

Let us show, in the case of A_n coefficients being nested sums of product of binomials, that the solution of the Calabi-Yau ODE, analytical at $x = 0$, which is by construction a series with integer coefficients, is actually a diagonal of rational function, and furthermore, that this rational function can actually be easily built.

5.1. Calculating the rational function for nested product of binomials

For pedagogical reasons we will just consider, here, a very simple example§ of a series $\mathcal{S}(x)$, with integer coefficients, given by a sum of product of binomials

$$\begin{aligned} \mathcal{S}(x) \;&=\; \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^3 \cdot x^n \\ &=\; HeunG(-8, -2, 1, 1, 1, 1,\; 8\,x) \;=\; HeunG(-1/8, 1/4, 1, 1, 1, 1, -x) \\ &=\; 1 \;+ 2\,x \;+ 10\,x^2 \;+ 56\,x^3 + 346\,x^4 \;+ 2252\,x^5 \;+ 15184\,x^6 \;+ 104960\,x^7 \\ &\qquad + 739162\,x^8 \;+ 5280932\,x^9 \;+ 38165260\,x^{10} \;+\; \cdots \end{aligned} \qquad (52)$$

This is the generating function of sequence **A** in Zagier's tables of binomial coefficients sums (see p. 354 in [96]).

The reader can easily get convinced that the calculations of this section can straightforwardly (sometimes tediously) be generalised to more complicated [97] nested sums of product of binomials†.

The diagonal of a rational function P/Q is written using Deligne's trick

$$\mathsf{Diag}\Bigl(\frac{P}{Q}\Bigr) \;=\; \Bigl(\frac{1}{2\,i\,\pi}\Bigr)^m \cdot \int_C \frac{P}{Q}\cdot\frac{dz_1}{z_1}\cdot\frac{dz_2}{z_2}\;\cdots\;\frac{dz_m}{z_m}, \qquad (53)$$

where C a vanishing† cycle [98], which is, with everyday words, the n-variables residue formula. Finding that a series is a diagonal of a rational function amounts to framing it into a residue form like (49). In order to achieve this, we write the binomial $\binom{n}{k}$ as the residue

$$\binom{n}{k} \;=\; \frac{1}{2\,i\,\pi}\cdot\int_C \frac{(1+z)^n}{z^k}\cdot\frac{dz}{z}, \qquad (54)$$

¶ The generating function of Harmonic numbers is $H(x) = \sum H_n \cdot x^n = -\ln(1-x)/(1-x)$.
§ See Proposition 7.3.2 in [90].
† Not necessarily corresponding to modular forms as can be seen on (81), (82).
† Cycle évanescent in french.

and, thus, we can rewrite $\mathcal{S}(x)$ as

$$(2\,i\,\pi)^3 \cdot \mathcal{S}(x) \;=$$

$$= \quad \sum_{n=0}^{\infty} \int\!\!\int\!\!\int \sum_{k=0}^{n} \frac{1}{(z_1\, z_2\, z_3)^k} \cdot \Bigl((1+z_1)\,(1+z_2)\,(1+z_3) \cdot \, x \Bigr)^n \cdot \frac{dz_1\, dz_2\, dz_3}{z_1\, z_2\, z_3}$$

$$= \int\!\!\int\!\!\int \sum_{n=0}^{\infty} \frac{1 \,-\, \Bigl(1/(z_1\, z_2\, z_3)\Bigr)^{(n+1)}}{1 \,-\, \Bigl(1/(z_1\, z_2\, z_3)\Bigr)} \cdot \Bigl((1+z_1)\,(1+z_2)\,(1+z_3) \cdot x \Bigr)^n \cdot \frac{dz_1\, dz_2\, dz_3}{z_1\, z_2\, z_3}$$

$$= \quad - \int\!\!\int\!\!\int \sum_{n=0}^{\infty} \frac{z_1\, z_2\, z_3}{1 \,-\, z_1\, z_2\, z_3} \cdot \Bigl((1+z_1)\,(1+z_2)\,(1+z_3) \cdot x \Bigr)^n \cdot \frac{dz_1\, dz_2\, dz_3}{z_1\, z_2\, z_3}$$

$$+ \int\!\!\int\!\!\int \sum_{n=0}^{\infty} \frac{1}{1 \,-\, z_1\, z_2\, z_3} \cdot \Bigl(\frac{(1+z_1)\,(1+z_2)\,(1+z_3) \cdot x}{z_1\, z_2\, z_3} \Bigr)^n \cdot \frac{dz_1\, dz_2\, dz_3}{z_1\, z_2\, z_3}$$

$$= \quad \int\!\!\int\!\!\int R(x;\, z_1,\, z_2,\, z_3) \cdot \frac{dz_1\, dz_2\, dz_3}{z_1\, z_2\, z_3}, \qquad (55)$$

where $R(x;\, z_1,\, z_2,\, z_3)$ reads:

$$\frac{z_1\, z_2\, z_3}{\bigl(1 \,-\, x \cdot\, (1+z_1)(1+z_2)(1+z_3)\bigr)\, \bigl(z_1\, z_2\, z_3 \,-\, x \cdot\, (1+z_1)(1+z_2)(1+z_3)\bigr)}.$$

From this last result one deduces immediately that (52) is actually the diagonal of:

$$\frac{1}{\bigl(1 \,-\, z_0 \cdot\, (1+z_1)(1+z_2)(1+z_3)\bigr) \cdot \bigl(1 \,-\, z_0\, z_1\, z_2\, z_3\, (1+z_1)(1+z_2)(1+z_3)\bigr)}.$$

Note that, as a consequence of a combinatorial identity due to Strehl and Schmidt [99, 100, 101], $\mathcal{S}(x)$ can also be written as

$$\mathcal{S}(x) \;=\; \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2\,k}{n} \cdot x^n \;=\; \sum_{n=0}^{\infty} \sum_{k=[n/2]-1}^{n} \binom{n}{k}^2 \binom{2\,k}{n} \cdot x^n. \qquad (56)$$

Calculations similar to (55) on this other binomial representation (56), enable to express (52) as the diagonal of an alternative rational function:

$$\frac{1}{\bigl(1 \,-\, z_0 \cdot\, (1+z_1)(1+z_2)(1+z_3)^2\bigr) \cdot \bigl(1 \,-\, z_0\, z_1\, z_2 \cdot\, (1+z_1)(1+z_2)\bigr)}. \qquad (57)$$

We thus see that we can actually *get explicitly*, from straightforward calculations, the rational function (56) for the Calabi-Yau-like ODEs (occurring from *differential geometry* or *enumerative combinatorics*) when series with nested sums of binomials take place, and, more generally, for enumerative combinatorics problems (related or not to Calabi-Yau manifolds) where series with *nested sums of binomials* take place.

Remark: These straightforward effective calculations guarantee to obtain an *explicit expression* for the rational function (56), however the rational function is far from being unique, and worse, the number of variables the rational function depends on is far from being the smallest possible number. Finding the "minimal" rational function (whatever the meaning of "minimal" may be) is a very difficult problem.

Recalling the well-known Apéry series $\mathcal{A}(x)$, and its rewriting due to Strehl and Schmidt [99, 100, 101],

$$\mathcal{A}(x) \, = \, \sum_{n=0}^{\infty} \sum_{k=0}^{n} {n \choose k}^2 {n+k \choose k}^2 \cdot x^n \, = \, \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} {n \choose k} {n+k \choose k} {k \choose j}^3 \cdot x^n$$
$$= \quad 1 \, + 5\,x \, + 73\,x^2 \, + 1445\,x^3 \, + 33001\,x^4 \, + \cdots, \qquad (58)$$

$\mathcal{A}(x)$ is known to be the diagonal of the rational function in five variables $1/R_1/R_2$ where R_1, R_2 read [66]:

$$R_1 \, = \, 1 \, - \, z_0, \qquad R_2 \, = \, (1 \, - \, z_1)(1 \, - \, z_2)(1 \, - \, z_3)(1 \, - \, z_4) \, - \, z_0 z_1 z_2,$$

as well as the diagonal of the rational function in five variables $1/Q_1/Q_2$ where Q_1, Q_2 read [67, 49]:

$$Q_1 \, = \, 1 \, - \, z_1\, z_2\, z_3\, z_4, \qquad Q_2 \, = \, (1 \, - \, z_3)(1 \, - \, z_4) \, - \, z_0 \cdot (1 \, + \, z_1)(1 \, + \, z_2),$$

and *also* the diagonal of the rational function in six variables $1/P_1/P_2/P_3$ where P_1, P_2, P_3 read [66]:

$$P_1 \, = \, 1 \, - z_0\, z_1, \qquad P_2 \, = \, 1 \, - z_2 \, - z_3 \, - z_0\, z_2\, z_3, \qquad P_3 \, = \, 1 \, - z_4 \, - z_5 \, - z_1\, z_4\, z_5.$$

A yet different diagonal representation for the Apéry series, due to Delaygue†, is provided by the diagonal of the rational function in eight variables:

$$\frac{1}{(1 \, - \, z_4 z_5 z_6 z_7) \cdot (1 \, - \, z_0 \cdot (1 + z_4)) \cdot (1 - z_1 \cdot (1 + z_5)) \cdot (1 - z_2 - z_6) \cdot (1 - z_3 - z_7)}.$$

Calculations similar to (55) on these new binomial expressions provides two new rational functions such that (58) can be written as the diagonal of one of these two rational functions. One is a rational function of five variables, of the form $1/Q_1^{(5)}/Q_2^{(5)}$

$$Q_1^{(5)} \, = \, 1 \, - z_0\, z_1\, z_2\, z_3\, z_4 \cdot (1 + z_1)\,(1 + z_2)\,(1 + z_3)\,(1 + z_4),$$
$$Q_2^{(5)} \, = \, 1 \, - z_0 \cdot (1 + z_1)\,(1 + z_2)\,(1 + z_3)^2\,(1 + z_4)^2, \qquad (59)$$

and the other one, is a rational function of six variables, of the form $1/Q_1^{(6)}/Q_2^{(6)}/Q_3^{(6)}$

$$Q_1^{(6)} \, = \, 1 \, - z_0\, z_3\, z_4\, z_5 \cdot (1 + z_1)\,(1 + z_2)^2\,(1 + z_3)\,(1 + z_4)\,(1 + z_5),$$
$$Q_2^{(6)} \, = \, 1 \, - z_0\, z_1\, z_2\, z_3\, z_4\, z_5 \cdot (1 + z_1)\,(1 + z_2),$$
$$Q_3^{(6)} \, = \, 1 \, - z_0 \cdot (1 + z_1)\,(1 + z_2)^2\,(1 + z_3)\,(1 + z_4)\,(1 + z_5). \qquad (60)$$

We thus see that, when a given function is a diagonal of a rational function, the rational function is far from being unique, the "simplest" representation (minimal number of variables, lowest degree polynomials, ...) being hard to find.

Similar computations show that the generating function of sequence **E** in Zagier's list [96]:

$$\mathcal{E}(x) \, = \, \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} 4^{n-2k} \cdot {n \choose 2k} {2k \choose k}^2 \cdot x^n$$
$$= \, 1 \, + 4\,x \, + 20\,x^2 \, + 112\,x^3 \, + 676\,x^4 \, + 4304\,x^5 \, + 28496\,x^6 \, + 194240\,x^7$$
$$+ 1353508\,x^8 \, + 9593104\,x^9 \, + 68906320\,x^{10} \, + \, \cdots \qquad (61)$$

† Private communication.

is the diagonal of the rational function in four variables

$$\frac{1}{(1-4z_0z_1z_2z_3\cdot(1+z_1))\cdot(1-z_0^2z_2z_3\cdot(1+z_2)^2(1+z_3)^2(1+z_1)^2)},$$

while the generating function of Zagier's sequence **B**

$$\begin{aligned}\mathcal{B}(x) \;&=\; \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k\cdot 3^{n-3k}\cdot\binom{n}{3k}\binom{3k}{k}\binom{2k}{k}\cdot x^n \\ &=\; 1\;+3\,x\;+9\,x^2\;+21\,x^3\;+9\,x^4\;-297\,x^5\;-2421\,x^6\;-12933\,x^7 \\ &\qquad -52407\,x^8\;-145293\,x^9\;-35091\,x^{10}\;+\;\cdots \end{aligned} \tag{62}$$

is the diagonal of the rational function in four variables

$$\frac{1}{(1\;-3\,z_0z_1z_2z_3\cdot(1+z_1))\cdot(1\;+z_0^3z_2^2z_3^2\cdot(1+z_1)^3(1+z_2)^3(1+z_3)^2)}. \tag{63}$$

Such calculations can systematically be performed on any series defined by nested sums of product of binomials. We have performed such calculations on a large number of the series corresponding to the list of Almkvist et al [91], that are given by such nested sums of product of binomials.

In fact, any s-nested sum of products of binomials raised to powers $\ell_1, \ldots, \ell_t$ can be written as the diagonal of a rational function in $\ell_1 + \cdots + \ell_t + 1$ variables, of the form $((1-Q_0)(1-Q_1)\cdots(1-Q_s))^{-1}$, where the Q_i's are products of powers of the variables z_i and of the linear forms $1+z_i$.

For instance, when $s = 1$, it is easy to prove using the same technique that the power series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\prod_{i=1}^{p} c^{an+bk}\cdot\binom{\alpha_i n\,+\beta_i k}{\gamma_i n+\delta_i k}\cdot x^n, \tag{64}$$

is the diagonal of the rational function in $p+1$ variables

$$\frac{1}{(1\;-c^a u\,\cdot\,z_0\,z_1\,\cdots\,z_p)\cdot(1\;-c^{a+b}\,\cdot\,u\,v\;z_0\,z_1\,\cdots\,z_p)}, \tag{65}$$

where u and b read:

$$u \;=\; \prod_{i=1}^{p}\frac{(1+z_i)^{\alpha_i}}{z_i^{\gamma_i}}, \qquad\qquad v \;=\; \prod_{i=1}^{p}\frac{(1+z_i)^{\beta_i}}{z_i^{\delta_i}}. \tag{66}$$

The same machinery provides in some cases diagonal representations for algebraic power series (as diagonals of bivariate rational functions) that are much simpler than those produced by Furstenberg's result sketched in Section 3.5. For instance, using the fact that

$$\binom{2n-2}{n-1} \;=\; \frac{1}{2\,i\,\pi}\cdot\int_C\frac{(1\,+z)^{2n-2}}{z^{n-1}}\cdot\frac{dz}{z}, \tag{67}$$

the algebraic function

$$f \;=\; \frac{z}{\sqrt{1-z}} \;=\; 4\cdot\sum_{n=0}^{\infty}\binom{2n-2}{n-1}\left(\frac{z}{4}\right)^n, \tag{68}$$

is readily seen to be the diagonal of the rational function

$$\frac{z_0\,z_1}{1\;-\;z_0\cdot(1\,+\,z_1)^2\,/4}, \tag{69}$$

which is much simpler than (25).

Similarly, the power series

$$\sum_{n=0}^{\infty} \binom{sn}{n} \cdot x^n, \tag{70}$$

is seen to be the diagonal of the rational function

$$\frac{1}{1 \, - z_0 \cdot \, (1 + z_1)^s}. \tag{71}$$

6. Comments and speculations

6.1. Christol's theorem

In [49] (page 61 Theorem 12, see also Proposition 7 in page 50 of [68]) it is proved that any power series with an *integral representation* (as defined in (72) see below) and of *maximal weight* for the corresponding *Picard-Fuchs linear differential equation* (denoted by L_V below) is the *diagonal of a rational function* and, in particular, is *globally bounded*.

The technical nature of the original papers is such that the result itself is difficult to find. This paragraph is devoted to explain, in down-to-earth terms, the somewhat esoteric expressions used in its wording, and to explain what it means on explicit examples. As the original proof is very obfuscated its principle is sketched in Appendix E.

A function f, analytic near 0, is said to have an "integral representation" if it can be written in the following form†:

$$f(x) \quad = \quad \int_C F(x; \, x_1, \, \dots, \, x_n) \cdot \, dx_1 \, \cdots \, dx_n, \tag{72}$$

where F is an *algebraic function*, hence living on some (projective) *complex* $(n+1)$-fold V, and C is a "cycle", namely an (oriented) compact (i.e. without boundary) *real* n-fold contained in V. In (72) x must be seen as a parameter. Then one integrates the n-differential $F(x; \dots) \, dx_1 \cdots dx_n$, that depends on x, on a "constant" cycle‡ C. If it exists, the integral representation is far from unique. Formula (49) shows that any diagonal of a rational function has an integral representation for which C is the so-called "vanishing [104] cycle". This is straightforwardly extended to diagonals of algebraic functions.

In practical examples, to obtain a cycle, one often needs to complete the integration domain by means of symmetries in the usual way when dealing with the method of residues. That happens for instance for the hypergeometric function $f(x) = B(a_2, \, b_1 \, - a_2) \cdot \, {}_2F_1([a_1, a_2], \, [b_1]; \, x)$ (B is the beta function, $\Re(b_1) \, > \, \Re(a_2) \, > \, 0$), which has the following Euler integral representation:

$$f(x) \quad = \quad \int_0^1 \, x_1^{a_2-1} \cdot \, (1 - x_1)^{b_1 - a_2 - 1} \, \cdot \, (1 \, - x \, x_1)^{-a_1} \, \cdot \, dx_1. \tag{73}$$

† Following [102] or, better, its version with parameter [103], one could define integral representations by integrating in (72) on domains C defined by polynomial (in $x_1, \, \dots, \, x_n$) equalities or inequalities. Actually the case $C = \{x_i \in [0, 1] \, ; \, 1 \, \le \, i \, \le \, n\}$ would be enough. To connect this definition with ours, one needs to use the Stokes theorem. However, the dimension n of the underlying complex manifold is basic for our next definitions and it is unfortunately not preserved by Stokes theorem.

‡ One could, more generally, integrate an m-differential for $m < n$. But Lefschetz theorems assert that, up to taking hyperplane sections, one can reduce to the case $n = m$.

More generally the only hypergeometric functions ${}_pF_q$ having an (Euler) integral representation are the ${}_{n+1}F_n$ with rational♯ parameters [105, 106] a_i, b_j:

$$ {}_{n+1}F_n([a_1, a_2, \cdots, a_n, a_{n+1}], [b_1, b_2, \cdots, b_n], x) $$
$$ = \quad \rho \cdot \int_0^1 \cdots \int_0^1 x_1^{a_1} \cdots x_n^{a_n} \cdot (1-x_1)^{b_1-a_1-1} \cdots (1-x_n)^{b_n-a_n-1} $$
$$ \times \, (1 - x_1 x_2 \cdots x_n \cdot x)^{-a_{n+1}} \cdot \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}. \qquad (74) $$

Moreover, the integration cycle C of (72) can involve points at infinity (we are dealing with projective geometry).

As C is a cycle, adding an exact differential to $F\, dx_1 \cdots dx_n$ clearly do not change f. But a famous Grothendieck's theorem [107] asserts that n-differentials on V up to exact ones built up a finite $\mathbb{C}(x)$-space. Moreover¶ differentiation under the integral endows this space with a connection, namely the Gauss-Manin one. In other words, f is solution of an ordinary linear differential equation (ODE) L_V, namely the *Picard-Fuchs differential equation*. This ODE depends only on V and do not involve any cycle but choosing a particular cycle C amount to choosing a particular solution f of L_V.

The Picard-Fuchs ODE L_V is known to have only regular singularities with rational exponents. Now L being a linear ODE for which 0 is a regular singularity, solutions near 0 of L are endowed with a "monodromy‡ weight": f is of weight W if L has $W+1$ solutions that are built in the following way:

$$ f(x) \, , \quad f(x) \cdot \log(x) \, + f_1(x) \, , \quad \cdots \, , \qquad (75) $$
$$ f(x) \cdot \frac{\log^W(x)}{W!} \; + f_1(x) \cdot \frac{\log^{W-1}(x)}{(W-1)!} \; + \; \cdots \; + \, f_W(x), $$

where the f_i are analytic near 0, and no solution involving $f(x) \cdot \log^{W+1}(x)$. For instance, a MUM (maximal unipotent monodromy) ODE L of order μ has a unique (up to a multiplicative constant) analytic solution near 0 and this solution is of logarithm weight $\mu - 1$.

Geometric considerations imply that, for solutions of L_V, the maximum monodromy weight is $n-1$. So we will say that f is of *maximal weight* for L_V if it is of weight $n-1$.

The Picard-Fuchs linear ODE is difficult to determine and, moreover, depends on the particular integral representation. What is well defined is the minimal linear ODE L_f of which f, with an integral representation, is solution. Then its Picard Fuchs ODE $L_V = M\, L_f$ is a left multiple of L_f and the monodromy weight of f for L_V is at least the monodromy weight of f for L_f (it is likely that the two monodromy weights are actually the same). The order of L_f is smaller, and often much smaller, than the order of L_V.

♯ The parameters have to be rational for the n-form in the integral representation (72) be algebraic. More deeply, when globally bounded, the hypergeometric function is a G-function and the corresponding linear differential operator is globally nilpotent hence it has only regular singularities with rational exponents. The parameters a_i and b_j are directly linked to exponents at 0 and ∞.
¶ See [108] specially § 1 and § 4 for instance but notice that the weight filtration used there and the monodromy weight we will use are quite distinct even if both constructions could look similar.
‡ The monodromy operator T, "turning once counterclockwise around 0", acts on the space S of solutions of L by $T(x) = x\, e^{2i\pi} = x$ and $T(\log(x)) = \log(x) + 2i\pi$. The solution f is of weight W if and only if it is in the image $(T-1)^W(S)$ but not in $(T-1)^{W+1}(S)$.

There is absolutely no reason for the order of L_f to be the number n of variables of the integral representation, but this effectively do happen in examples, notably for hypergeometric functions ${}_{n+1}F_n$ and for certain Calabi-Yau linear ODE. Under these circumstances, if L_f is MUM then f is of maximum weight for L_V and the theorem asserts that f is the diagonal of a rational function. Let us remark that in that case f is the unique (up to a multiplicative constant) analytic solution near 0 of L_f.

Disappointingly, when this result can be applied to ${}_{n+1}F_n$, it becomes somewhat trivial. More precisely, the hypergeometric function is of maximal weight if and only if $b_j = 1$ for all j (there is only $n!$'s in the denominator of coefficients). In that case it is obviously the Hadamard product of algebraic functions:

$$\begin{aligned} &{}_nF_{n-1}([\alpha_1,\, \alpha_2,\, \cdots,\, \alpha_n],\, [1,\, 1,\, \cdots\, 1],\, x) \\ &\quad = \quad (1-x)^{-\alpha_1} \star (1-x)^{-\alpha_2} \;\cdots\, \star\, (1-x)^{-\alpha_n}. \end{aligned} \tag{76}$$

Therefore, we now have (at least) three sets of problems yielding diagonal of rational functions: the n-fold integrals of the form (49) with (50), the Picard-Fuchs linear ODEs with solution of maximal monodromy weight and, finally, the problems of enumerative combinatorics where nested sums of products of binomials take place. Diagonal of rational functions, thus, occur in a quite large set of problems of theoretical physics.

6.2. Christol's conjecture

The diagonal of a rational function is globally bounded (i.e. it has non zero radius of convergence and integer coefficients up to one rescaling) and *D-finite* (i.e. solution of a linear differential equation with polynomial coefficients)‡.

The reciprocal statement is the "Christol's conjecture" [49] saying that any D-finite, globally bounded series is necessarily the diagonal of a rational function.

A fantastic Chudnovski theorem ([110] page 267) asserts that the minimal linear differential operator of a G function (and in particular of a D-finite globally bounded series) is a G-operator (i.e. at least conjecturally, a globally nilpotent operator) [28, 29, 30]. "Christol's conjecture" amounts to saying something more: if the solution of this globally nilpotent linear differential operator is, not only a G-series, but a *globally bounded series, then it is the diagonal of a rational function.*

Conversely the solution, analytical at 0, of a globally nilpotent linear differential operator is necessarily a G-function [29, 30]. Moreover, a "classical" conjecture, with numerous avatars, claims that any G-function comes from geometry i.e. roughly speaking, it has an integral representation§.

To test the validity of Christol's conjecture we look for counter-examples not contradicting classical conjectures. Then we search D-finite power series with integer coefficients which are not algebraic but have an integral representation and are not of maximal weight for the corresponding Picard-Fuchs linear ODE.

As a first step let us limit ourself to hypergeometric functions ${}_{n+1}F_n$. The monodromy weight W is exactly the number of 1 among the b_i.

When ${}_{n+1}F_n$ is globally bounded and has no integer parameters b_i ($W = 0$), its minimal ODE has a p-curvature zero for almost all primes p. However, a Grothendieck

‡ The series expansion of the susceptibility of the isotropic 2-D Ising model can be recast into a series with integer coefficients (see [15, 27, 109, 19]), but it cannot be the diagonal of rational functions since the full susceptibility is *not a D-finite function* [109].

§ Bombieri-Dwork conjecture see for instance [30].

conjecture, proved for ${}_3F_2$ in [108], and generalised to ${}_{n+1}F_n$ in [111], asserts that, under these circumstances, the hypergeometric function is *algebraic*.

So we are looking for *globally bounded* hypergeometric functions satisfying $1 \le W \le n-1$. In general such hypergeometric functions are G-series but are very far from being globally bounded. The hypergeometric world extends largely outside the world of diagonal of rational functions.

Such an example in the first case $n = 2$, $W = 1$ was given in [49]:

$$ {}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right], \left[\frac{1}{3}, 1\right], 3^6\, x\right) \quad = \quad 1 \ + 60\, x \ + 20475\, x^2 \ + 9373650\, x^3 $$
$$ + 4881796920\, x^4 \ + 2734407111744\, x^5 \ + 1605040007778900\, x^6 \ + \ \cdots \qquad (77) $$

The integer coefficients read with the rising factorial (or Pochhammer) symbol

$$ \frac{(1/9)_n \cdot (4/9)_n \cdot (5/9)_n}{(1/3)_n \cdot (1)_n \cdot n!} \cdot 3^{6n} \quad = \quad \frac{\rho(n)}{\rho(0)}, \qquad (78) $$

where:

$$ \rho(n) \quad = \quad \frac{\Gamma(1/9+n)\,\Gamma(4/9+n)\,\Gamma(5/9+n)}{\Gamma(1/3+n)\,\Gamma(1+n)\,\Gamma(1+n)} \cdot 3^{6n}. \qquad (79) $$

Note that, at first sight, it is *far from clear*§ on (79), or on the simple recursion on the $\rho(n)$ coefficients (with the initial value $\rho(0) = 1$)

$$ \frac{\rho(n+1)}{\rho(n)} \quad = \quad 3 \cdot \frac{(1+9n)\,(4+9n)\,(5+9n)}{(1+3n)(1+n)^2}, \qquad (80) $$

to see that the $\rho(n)$'s are actually integers. A sketch of the (quite arithmetic) proof that the $\rho(n)$'s are actually integers, is given in Appendix G.

Because of the $1/3$ in the right (lower) parameters of (77), the hypergeometric function (77) is not an obvious Hadamard product of algebraic functions (and thus a diagonal of a rational function), and one can see that it is not an algebraic hypergeometric function either by calculating its p-curvature and finding that it is not zero [93], or using [112]. Proving that an algebraic function is the diagonal of a rational function and proving that a solution of maximal weight for a Picard-Fuchs equation is the diagonal of a rational function use two entirely distinct ways. The hope is to combine both techniques to conclude in the intermediate situation.

This example remained for twenty years, the only "blind spot" on Christol's conjecture. We have recently found many other ${}_3F_2$ examples‡, such that their series expansion have *integer coefficients* but are not obviously diagonal of rational functions. These new hypergeometric examples are displayed in Appendix F. Unfortunately these hypergeometric examples are on the same "frustrating footing" as Christol's example (77): we are not able to show that one of them is actually a diagonal of a rational function, or, conversely, to show that one of them cannot be the diagonal of a rational function.

7. Integrality versus modularity: learning by examples

A large number of examples of integrality of series-solutions comes from modular forms. Let us just display two such modular forms associated with HeunG functions of the form $HeunG(a, q, 1, 1, 1, 1; x)$.

§ In contrast with cases where binomial (and thus integers) expressions take place.

‡ ${}_2F_1$ cases are straightforward, and cannot provide counterexamples to Christol's conjecture.

7.1. First modular form example

One can, for instance, rewrite the example (52) of subsection (5.1), namely $HeunG(-1/8, 1/4, 1, 1, 1, 1, -x)$, as a hypergeometric function† with *two rational pullbacks*:

$$HeunG(-1/8, 1/4, 1, 1, 1, 1, -x) \quad = \quad \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^3 x^n \qquad (81)$$

$$= \quad \Bigl((1 + 4x) \cdot (1 + 228x + 48x^2 + 64x^3)\Bigr)^{-1/4}$$

$$\times \ {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1]; \frac{1728 \cdot (1 - 8x)^6 \cdot (1 + x)^3 \cdot x}{(1 + 228x + 48x^2 + 64x^3)^3 \cdot (1 + 4x)^3}\Bigr)$$

$$= \quad \Bigl((1 - 2x) \cdot (1 - 6x + 228x^2 - 8x^3)\Bigr)^{-1/4}$$

$$\times \ {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1]; \frac{1728 \cdot (1 - 8x)^3 \cdot (1 + x)^6 \cdot x^2}{(1 - 2x)^3 \cdot (1 - 6x + 228x^2 - 8x^3)^3}\Bigr).$$

The relation between the two pullbacks, that are related by the "Atkin¶" involution§ $x \leftrightarrow -1/8/x$, gives the modular curve:

$$1953125\, y^3 z^3 \ - 187500\, y^2 z^2 \cdot (y + z) \ + 375\, y z \cdot (16 z^2 - 4027\, z y + 16 y^2)$$
$$- 64 \cdot (z + y) \cdot (y^2 + z^2 + 1487\, z y) \ + 110592 \cdot z y \quad = \quad 0. \qquad (82)$$

Series (52) is solution of the (exactly) *self adjoint* linear differential operator Ω where ($\theta = x \cdot D_x$):

$$x \cdot \Omega \quad = \quad \theta^2 \ - x \cdot (7\theta^2 + 7\theta + 2) \ - 8x^2 \cdot (\theta + 1)^2. \qquad (83)$$

The relevance of the "Atkin" involution $x \leftrightarrow -1/8/x$ is also clear on the operator: note that operator Ω is invariant by changing $\theta \leftrightarrow -1 - \theta$, and $x \leftrightarrow -1/8/x$. Also note that changing Ω by a pullback $x \leftrightarrow -1/8/x$, amounts to changing Ω into $x \cdot \Omega \cdot x^{-1}$.

7.1.1. Modular invariance

Do note that these pullbacks are respectively of the form ($\circ$ denotes the composition of functions):

$$\mathcal{M}_2 \ = \ \frac{1728\, x}{(x + 16)^3} \circ \frac{(1 - 8x)^3}{x\,(1 + x)^3} \quad \text{and:} \quad \tilde{\mathcal{M}}_2 \ = \ \frac{1728\, x}{(x + 256)^3} \circ \frac{(1 - 8x)^3}{x\,(1 + x)^3}, \qquad (84)$$

† The relation between modular forms and hypergeometric functions can be simply seen in the identity ${}_2F_1([1/12, 5/12], [1], 1728/j(\tau))^4 \ = \ E_4(\tau)$, where E_4 is an Eisenstein series.

¶ In previous papers [113, 32], with some abuse of language, we called such an involution an *Atkin-Lehner involution*. In fact this terminology is commonly used in the mathematical community for an involution $\tau \rightarrow -N/\tau$, on τ, the ratio of periods, and *not* for our x-involution. However, *when the modular curve is of genus zero* one has a parametrisation in term of the variable $x_N(\tau) = (\eta(\tau)/\eta(N\tau))^{24/(N-1)}$ (see eq. (27) in [114]), which actually transforms *as an involution* ($x_N \rightarrow A/x_N$) under the Atkin-Lehner involution. This is why we switch to the wording *"Atkin" involution*.

§ The relevance of the "Atkin" involution $x \leftrightarrow -1/8/x$ is also clear on the operator: note that operator Ω is invariant by changing $\theta \leftrightarrow -1 - \theta$, and $x \leftrightarrow -1/8/x$. Also note that changing Ω by a pullback $x \leftrightarrow -1/8/x$, amounts to changing Ω into $x \cdot \Omega \cdot x^{-1}$.

where one recognises the two Hauptmoduls of the modular curve corresponding to $\tau \, \rightarrow \, 2\,\tau$. Introducing the Dedekind-like parametrisation:

$$x(q) \, = \, q \cdot \prod_{n=1}^{\infty} (1 \, - \, q^n)^{a(n)} \qquad \hbox{where:} \qquad \sum_{n=1}^{\infty} a(n) \cdot t^n \, = \, {{3\,t \cdot \, (1 \, - t^2)} \over {1 \, - \, t^6}},$$

one can rewrite the Hauptmoduls as:

$$\tilde{\mathcal{M}}_2\Bigl({{(1 \, - 8\,x(q))^3} \over {x(q) \cdot \, (1 \, + \, x(q))^3}} \Bigr) \, = \, \mathcal{M}_2\Bigl({{(1 \, - 8\,x(q^2))^3} \over {x(q^2) \cdot \, (1 \, + \, x(q^2))^3}} \Bigr). \tag{85}$$

7.1.2. Other representations

In fact, using Kummer's relation and other relations on ${}_2F_1$'s:

$$\begin{aligned} {}_2F_1\Bigl([{{1} \over {6}}, \, {{1} \over {3}}], \, [1], \, 4\,x\,(1-x)\Bigr) \, &= \, {}_2F_1\Bigl([{{1} \over {3}}, \, {{2} \over {3}}], \, [1], \, x\Bigr) \\ &= \, (1-x)^{-1/3} \cdot \, {}_2F_1\Bigl([{{1} \over {3}}, \, {{1} \over {3}}], \, [1], \, -{{x} \over {1-x}}\Bigr) \\ &= \, (1-x)^{-2/3} \cdot \, {}_2F_1\Bigl([{{2} \over {3}}, \, {{2} \over {3}}], \, [1], \, -{{x} \over {1-x}}\Bigr) \\ &= \, \Bigl({{9} \over {9 \, - \, 8\,x}}\Bigr)^{1/4} \cdot \, {}_2F_1\Bigl([{{1} \over {12}}, \, {{5} \over {12}}], \, [1], \, {{64\,x^3 \cdot \, (1-x)} \over {(9 \, - \, 8\,x)^3}}\Bigr). \end{aligned} \tag{86}$$

$HeunG(-1/8, 1/4, 1, 1, 1, 1, -x)$ can be written in many different ways as hypergeometric ${}_2F_1$ with *two* pullbacks:

$$\begin{aligned} HeunG(-1/8, 1/4, 1, 1, 1, 1, -x) \, &= \, {{1} \over {1 \, + \, 4\,x}} \cdot \, {}_2F_1\Bigl([{{1} \over {3}}, \, {{2} \over {3}}], \, [1], \, {{27\,x} \over {(1 \, + \, 4\,x)^3}}\Bigr) \\ &= \, {{1} \over {1 \, - \, 2\,x}} \cdot \, {}_2F_1\Bigl([{{1} \over {3}}, \, {{2} \over {3}}], \, [1], \, {{27\,x^2} \over {(1 \, - \, 2\,x)^3}}\Bigr) \\ &= \, (1 \, + x)^{-1/3} \cdot (1 \, - 8\,x)^{-2/3} \cdot \, {}_2F_1\Bigl([{{1} \over {3}}, \, {{1} \over {3}}], \, [1], \, - \, {{27\,x} \over {(1 \, + \, x) \cdot (1 - 8\,x)^2}}\Bigr) \\ &= \, {{1} \over {1 \, + \, 4\,x}} \cdot \, {}_2F_1\Bigl([{{1} \over {6}}, \, {{1} \over {3}}], \, [1], \, {{108\,x\,(1+x)\,(1 \, - 8\,x)^2} \over {(1 \, + \, 4\,x)^6}}\Bigr) \, = \, \cdots \end{aligned} \tag{87}$$

The relation between the two pullbacks in (87), namely $u \, = \, 27\,x/(1+\,4\,x)^3$, and $v \, = \, 27\,x^2/(1-\,2\,x)^3)$, is the genus-zero modular curve:

$$\begin{aligned} &8\,u^3v^3 \, - 12\,u^2v^2 \cdot \, (u+v) \, + 3\,uv \cdot \, (2\,u^2 + 2\,v^2 + 13\,uv) \\ &\quad - \, (u+v) \cdot \, (v^2 + 29\,uv + u^2) \, + 27\,uv \, = \, 0. \end{aligned} \tag{88}$$

Let us consider the modular curve (88) for $u \, = x$, seeing v as a algebraic function of x. One of the three root-solutions expands as:

$$\tilde{v}_0(x) \, = \, {{1} \over {27}}\,x^2 \, + {{10} \over {243}}\,x^3 + {{256} \over {6561}}\,x^4 + {{18928} \over {531441}}\,x^5 + {{154000} \over {4782969}}\,x^6 \, + \, \cdots, \tag{89}$$

the two other ones being Puiseux series (here t denotes $x^{1/2}$):

$$\tilde{v}_1(x) \, = \, 3\,\sqrt{3}\,t \, - 15\,t^2 + {{119} \over {6}}\sqrt{3}\,t^3 - {{1904} \over {27}}\,t^4 + {{50701} \over {648}}\sqrt{3}\,t^5 \, + \, \cdots, \tag{90}$$

the third root, $\tilde{v}_2(x)$ corresponding to change t into $-t$. These series expansions compose nicely: $\tilde{v}_0(\tilde{v}_1(x)) \, = \, \tilde{v}_0(\tilde{v}_2(x)) \, = \, x$.

Remark: Ramanujan's cubic transformation. For the hypergeometric function ${}_2F_1([1/3, 2/3], [1], x)$, the existence of *two* pullbacks is also reminiscent of relation (28) in [33], or to the known Ramanujan's cubic transformation formula [115]:

$$(1 \, + \, 6\,x) \cdot {}_2F_1\Big([\frac{1}{3}, \, \frac{2}{3}], \, [1], \, 27\,x^3\Big) \quad = \quad {}_2F_1\Big([\frac{1}{3}, \, \frac{2}{3}], \, [1], \, 1 \, - \, \Big(\frac{1 - 3\,x}{1 \, + \, 6\,x}\Big)^3\Big)$$
$$= \quad 1 \quad + 6\,x \quad + 6\,x^3 \quad + 36\,x^4 \quad + 90\,x^6 \quad + 540\,x^7 \, + 1680\,x^9 \quad + \, \cdots \qquad (91)$$

Noting that ${}_2F_1\Big([1/3, \, 2/3], \, [1], \, x)$ and ${}_2F_1\Big([1/3, \, 2/3], \, [1], \, 1 - x)$ are solutions of the *same* second order linear ODE, one can change one of the two pullbacks, $P_1 \, = \, 27\,x^3$ and $P_2 \, = \, 1 \, - \, (1 \, - \, 3\,x)^3/(1 \, + \, 6\,x)^3$ in (91), into $1 \, - \, P_i, \; i \, = \, 1, 2$. The relation between the two pullbacks $u \, = \, 1 \, - \, P_1$ and $v \, = \, P_2$, is a simple (genus zero, (u, v)-symmetric) modular curve:

$$512 \cdot \, u^3 \cdot \, v^3 \; - 1728 \cdot \, u^2 \cdot \, v^2 \cdot \, (u + v) \quad + 216 \cdot \, u\,v \cdot \, (17 \cdot \, u\,v + 9 \cdot \, (u^2 + v^2))$$
$$- \, 243 \cdot \, (v + u) \cdot \, (3 \cdot \, (u^2 + v^2) \quad + 8 \cdot \, u \, v) \quad + 729 \cdot \, (u^2 + u\,v + v^2) \quad = \quad 0. \qquad (92)$$

7.1.3. Schwarzian condition

A necessary condition for *two different* rational (resp. algebraic) pullbacks to exist for a hypergeometric function like ${}_2F_1([1/3, \, 2/3], \, [1], \, x)$, i.e. a necessary condition for a relation

$${}_2F_1\Big([\frac{1}{3}, \, \frac{2}{3}], \, [1], \, p_1(x)\Big) \quad = \quad r_{1,2}(x) \cdot {}_2F_1\Big([\frac{1}{3}, \, \frac{2}{3}], \, [1], \, p_2(x)\Big), \qquad (93)$$

for some algebraic functions $r_{1,2}(x)$, is the (symmetric) condition

$$\{p_1(x), \, x\} \quad + \quad \frac{1}{18} \cdot \frac{8\,p_1(x)^2 \; - 8\,p_1(x) \; + 9}{p_1(x)^2 \cdot (p_1(x) - 1)^2} \cdot \Big(\frac{dp_1(x)}{dx}\Big)^2$$
$$= \quad \{p_2(x), \, x\} \quad + \quad \frac{1}{18} \cdot \frac{8\,p_2(x)^2 \; - 8\,p_2(x) \; + 9}{p_2(x)^2 \cdot (p_2(x) - 1)^2} \cdot \Big(\frac{dp_2(x)}{dx}\Big)^2, \qquad (94)$$

where $\{p_1(x), \, x\}$ denotes the *Schwarzian derivative*. This condition is invariant by the simple transformations $p_i \, \rightarrow \, 1 \, - \, p_i, \; i \, = \, 1, 2$. One immediately verifies that the *Schwarzian condition* (94) is actually verified for the pullbacks $(p_1(x), \, p_2(x))$ occurring in (91), namely $(27\,x^3, \; 1 \, - \, (1 \, - \, 3\,x)^3/(1 \, + \, 6\,x)^3)$, or in (87), namely $(27\,x/(1 + \, 4\,x)^3, \, 27\,x^2/(1 - \, 2\,x)^3)$.

Let us consider the modular curve (92) for $u \, = x$, seeing v as a algebraic function of x. The three root-solutions expand as:

$$v_0(x) \; = \quad 1 \quad - \frac{1}{729}\,x^3 \quad - \frac{5}{2187}\,x^4 \quad - \frac{56}{19683}\,x^5 \quad - \frac{1691}{531441}\,x^6 \quad - \frac{5390}{1594323}\,x^7 \quad + \quad \cdots \, ,$$

$$v_1(x) \; = \; \left(\frac{i\sqrt{3}}{2} - \frac{1}{2}\right) \cdot \, x \, + \, \frac{5\,i\sqrt{3}}{9}\,x^2 \; + \left(\frac{i\sqrt{3}}{2} + \frac{19}{54}\right) \cdot \, x^3 \, + \left(\frac{65}{162}\,i\sqrt{3} + \frac{95}{162}\right) \cdot \, x^4$$
$$+ \left(\frac{70}{243}\,i\sqrt{3} + \frac{532}{729}\right) \cdot \, x^5 \; + \left(\frac{1171}{1458} + \frac{2297}{13122}\,i\sqrt{3}\right) \cdot \, x^6 \quad + \quad \cdots \, , \qquad (95)$$

the third series expansion $v_2(x)$ having its coefficients complex conjugate of the ones in the series expansion v_1. These series expansions compose nicely:

$$v_1(v_2(x)) \; = \; v_2(v_1(x)) \; = \; x, \qquad v_0(v_1(x)) \; = \; v_0(v_2(x)) \; = \; v_0(x), \qquad (96)$$
$$v_1(v_1(x)) \; = \; v_2(x), \quad v_2(v_2(x)) \; = \; v_1(x), \quad v_1(v_1(v_1(x))) \; = \; v_2(v_2(v_2(x))) \; = \; x.$$

One verifies on these three series expansions (95) the condition corresponding to the $p_1(x) \, = \, x$ subcase in (94):

$$\{v(x), \, x\} \, + \, {{1} \over {18}} \cdot {{8 \, v(x)^2 \, - 8 \, v(x) \, + 9} \over {v(x)^2 \cdot (v(x)-1)^2}} \cdot \Bigl({{dv(x)} \over {dx}}\Bigr)^2 \, = \, {{1} \over {18}} \cdot {{8 \, x^2 \, - 8 \, x \, + 9} \over {x^2 \cdot (x-1)^2}}. \quad (97)$$

One also verifies that the three series expansions (89) and (90) also satisfy, as they should, the previous Schwarzian condition (97). One also verifies, as it should, that the composition of all these series satisfy (97). For instance

$$\tilde{v}_0(\tilde{v}_0(x)) \, = \, {{1} \over {19683}} \, x^4 \, + {{20} \over {177147}} \, x^5 \, + {{274} \over {1594323}} \, x^6 \, + {{86636} \over {387420489}} \, x^7 \, + \, \cdots,$$
$$v_0(\tilde{v}_0(x)) \, = \, 1 \, - {{1} \over {14348907}} \, x^6 \, - {{10} \over {43046721}} \, x^7 \, - {{187} \over {387420489}} \, x^8 \, + \, \cdots, \quad (98)$$

satisfies (97).

In other words, the *Schwarzian condition* (94) is another way to encode the modular curves (92) or (88), and in fact, an infinite number of modular curves. The emergence of Schwarzian derivatives should not be seen as a surprise. A relation like ${}_2F_1([1/3, \, 2/3], [1], \, v(x)) \, = \, R_{1,2}(x) \cdot \, {}_2F_1([1/3, 2/3], [1], \, x)$ is obviously stable by the composition† of the pullback functions $v(x)$.

7.2. Second modular form example

The integrality of series-solutions can be quite non-trivial like the solution of the Apéry-like operator

$$\Omega \, = \, x \cdot (1 \, - 11 \, x \, - x^2) \cdot D_x^2 \, + (1 \, - 22 \, x \, - 3 \, x^2) \cdot D_x \, - (x+3), \quad (99)$$
$$\text{or:} \quad x \cdot \Omega \, = \, \theta^2 \, - x \cdot \, (11 \, \theta^2 \, + 11 \, \theta \, + 3) \, - x^2 \cdot \, (\theta \, + 1)^2,$$

which can be written as a HeunG function. Introducing $\alpha \, = \, 11/2 - 5 \cdot 5^{1/2}/2$, this (at first sight involved) HeunG function reads:

$$HeunG\Bigl(-{{123} \over {2}} \, + {{55} \over {2}} \cdot 5^{1/2}, \, -{{33} \over {2}} \, + {{15} \over {2}} \cdot 5^{1/2}, 1, 1, 1, 1, \, \alpha \cdot x\Bigr)$$
$$= \, {{1} \over {1 - \alpha \, x}} \cdot \, HeunG\Bigl({{1} \over {2}} \, - {{11} \over {50}} \cdot 5^{1/2}, \, {{1} \over {2}} \, - {{1} \over {10}} \cdot 5^{1/2}, 1, 1, 1, 1, \, - {{\alpha \, x} \over {1 - \, \alpha \, x}}\Bigr)$$
$$= \, \sum_{n=0}^{\infty} \, \sum_{k=0}^{n} \, {{n} \choose {k}}^2 \, {{n+k} \choose {k}} \cdot \, x^n$$
$$= \, 1 \, + 3 \cdot x \, + 19 \cdot x^2 \, + 147 \cdot x^3 \, + 1251 \cdot x^4 \, + 11253 \cdot x^5 \, + 104959 \cdot x^6$$
$$+ 1004307 \cdot x^7 + 9793891 \cdot x^8 \, + 96918753 \cdot x^9 \, + 970336269 \cdot x^{10}$$
$$+ 9807518757 \cdot x^{11} \, + 99912156111 \cdot x^{12} \, + 1024622952993 \cdot x^{13}$$
$$+ 10567623342519 \cdot x^{14} \, + 109527728400147 \cdot x^{15} \, + 1140076177397091 \cdot x^{16}$$
$$+ 11911997404064793 \cdot x^{17} \, + 124879633548031009 \cdot x^{18}$$
$$+ 1313106114867738897 \cdot x^{19} \, + \, \cdots \quad (100)$$

† The Schwarzian derivative $S(f) \, = \, \{f(x), \, x\}$ is the well-suited derivative to take into account when composition of functions occurs. This can, for instance, be seen on the chain rule: $(S(f \circ g))(z) \, = \, S(f)(g(z)) \cdot g'(z)^2 \, + S(g)$.

but *actually corresponds to a modular form*, which can be written in two different ways using *two pullbacks*:

$$
\begin{aligned}
&(x^4 + 12\,x^3 + 14\,x^2 - 12\,x + 1)^{-1/4} \\
&\qquad \times\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1];\, \frac{1728 \cdot x^5 \cdot (1 - 11\,x - x^2)}{(x^4 + 12\,x^3 + 14\,x^2 - 12\,x + 1)^3}\Big) \\
&= \quad (1 + 228\,x + 494\,x^2 - 228\,x^3 + x^4)^{-1/4} \qquad\qquad (101) \\
&\qquad \times\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1];\, \frac{1728 \cdot x \cdot (1 - 11\,x - x^2)^5}{(1 + 228\,x + 494\,x^2 - 228\,x^3 + x^4)^3}\Big).
\end{aligned}
$$

Do note that these two pullbacks are respectively of the form:

$$
\begin{aligned}
&\frac{1728 \cdot x^5 \cdot (1 - 11\,x - x^2)}{(x^4 + 12\,x^3 + 14\,x^2 - 12\,x + 1)^3} \qquad\qquad (102) \\
&\qquad = \quad \frac{1728 \cdot x}{(x^2 + 10\,x + 5)^3} \circ \frac{1 - 11\,x - x^2}{x},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1728 \cdot x \cdot (1 - 11\,x - x^2)^5}{(1 + 228\,x + 494\,x^2 - 228\,x^3 + x^4)^3} \qquad\qquad (103) \\
&\qquad = \quad \frac{1728 \cdot x^5}{(x^2 + 250\,x + 3125)^3} \circ \frac{1 - 11\,x - x^2}{x},
\end{aligned}
$$

where one recognises [116] the two Hauptmoduls of the modular curve corresponding to $\tau \rightarrow 5 \cdot \tau$:

$$
\begin{aligned}
\mathcal{M}_5(x) \,&=\, \frac{1728 \cdot x}{(x^2 + 10\,x + 5)^3}, \qquad\qquad (104) \\
\tilde{\mathcal{M}}_5(x) \,&=\, \frac{1728 \cdot x^5}{(x^2 + 250\,x + 3125)^3} \,=\, \mathcal{M}_5\left(\frac{5^3}{x}\right).
\end{aligned}
$$

Also note that the fact that the relation between the two Hauptmoduls (104) corresponds to $\tau \rightarrow 5 \cdot \tau$ can be seen straightforwardly if one introduces the quite non-trivial Dedekind-like parametrisation

$$
x(q) \,=\, q \cdot \prod_{n=1}^{\infty} (1 - q^n)^{a(n)}, \quad \text{where:} \quad \sum_{n=1}^{\infty} a(n) \cdot t^n \,=\, \frac{5 \cdot t \cdot (1 - t)\,(1 - t^2)}{1 - t^5},
$$

yielding:

$$
\tilde{\mathcal{M}}_5\Big(\frac{1 - 11\,x(q^5) - x^2(q^5)}{x(q^5)}\Big) \,=\, \mathcal{M}_5\Big(\frac{1 - 11\,x(q) - x^2(q)}{x(q)}\Big). \qquad (105)
$$

More modular form examples of series with integer coefficients are given in Appendix H. The modular form examples displayed in Appendix A, corresponded to lattice Green functions [52]. Therefore, they have *n-fold integral representations*, and, *after* section (4.4), can be seen to *be diagonals of rational functions*. In contrast the modular form examples displayed in Appendix H correspond to differential geometry examples discovered by Golyshev and Stienstra [117], where no n-fold integral representation is available at first sight.

7.3. Remark

At first sight one might think, and this is almost suggested in the literature, that the integrality of the series-solutions of the globally nilpotent operators, corresponds to some deep arithmetic property and, in the same time, that these integer coefficients have a deep "physical" meaning (instantons, ...). There is often some confusion in the literature between the concept of *integrality* of series (globally bounded series) and the concept of *modularity* [34] which suggests a connection with selected algebraic varieties (modular forms and elliptic functions [118], mirror maps, Calabi-Yau manifolds).

Note that, along this "selected algebraic varieties" line, it is also worth recalling Krammer-Deitweiler's counter-example [119, 120] to Dwork's conjecture of a globally nilpotent operator that cannot be reduced to an operator having hypergeometric solutions up to a pullback, which corresponds to the two following HeunG functions:

$$HeunG(81,\, 1/2,\, 1/6,\, 1/3,\, 1/2,\, 1/2,\, 81\,x), \qquad (106)$$
$$x^{1/2} \cdot HeunG(81,\, 21,\, 2/3,\, 5/6,\, 3/2,\, 1/2,\, 81\,x).$$

These two HeunG are solutions of a globally nilpotent linear differential operator† but *are not globally bounded*: they are G-series that *cannot* be reduced to series with *integer coefficients*. However, this special example corresponds to a special *arithmetic* situation: we are dealing, here, with very special HeunG functions associated with periods of a family of *abelian surfaces over a Shimura curve* [120], and very special ODEs corresponding to uniformising linear differential equations of *arithmetic Fuchsian lattices*.

The interest of demonstration of sections (4.2), (4.4) is to show that this integrality can be, in fact, the straight consequence of a quite simple form of the n-fold integral namely (49) with the analyticity condition (50), that can be expected to be seen in a quite general framework (for instance in enumerative combinatorics), far from the theory of elliptic curves, or the theory of Calabi-Yau manifolds and other Frobenius manifolds [121]. Let us now try to "disentangle" the concept of *integrality* of series (globally bounded series) and the concept of *modularity*.

8. Integrality versus modularity

8.1. Diffeomorphisms of unity pullbacks

Let us consider a first simple example of a hypergeometric function which is solution of a Calabi-Yau ODE, and which occurred, at least two times in the study of the Ising susceptibility n-fold integrals [32, 33] $\chi^{(n)}$ and $\chi_d^{(n)}$, namely ${}_4F_3([1/2,\, 1/2,\, 1/2,\, 1/2],\, [1,\, 1,\, 1], 256\,x)$, where we perform a (diffeomorphism of unity) pullback:

$$ {}_4F_3\Big([\frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2}],\, [1,\, 1,\, 1],\;\; \frac{256\;x}{1\, +c_1\,x\, +c_2\,x^2\, +\,\cdots}\Big) \;\; = \;\; 1 \;\; +16\cdot x \qquad (107)$$
$$ +\,(1296 - 16\,c_1)\cdot x^2 \;\; +\,(160000 + 16\,c_1^2 \,-16\,c_2 \,-2592\,c_1)\cdot x^3 \;\; + \;\cdots$$

If the pull-back in (107) is such that the coefficients c_n, at its denominator, are integers, one finds that the series expansion is actually a series with integer coefficients, for *every such pullback* (i.e. for every integer coefficients c_n). Furthermore, a

† Generically HeunG functions with rational parameters are not globally nilpotent, which indicates that they do not have an integral representation as an integral of some algebraic integrand.

straightforward calculation of the corresponding nome $q(x)$ and its compositional inverse (mirror map) $x(q)$, *also yields series with integer coefficients*:

$$q(x) \;=\; x \;+(64-c_1)\cdot x^2 \;+(c_1^2+7072-c_2-128\,c_1)\cdot x^3 \;+\;\cdots, \qquad (108)$$

$$x(q) \;=\; q \;+(c_1-64)\cdot q^2 \;+(c_1^2+1120+c_2-128\,c_1)\cdot q^3 \;+\;\cdots, \qquad (109)$$

when its Yukawa coupling [32], seen as a function of the nome q, $K(q)$ is also a series with integer coefficients and is *independent of the pullback*:

$$K(q) \;=\; 1 \;+32\cdot q \;+4896\cdot q^2 \;+702464\cdot q^3 \;+\cdots \qquad (110)$$

This independence of the Yukawa coupling with regards to pullbacks, is a known property, and has been proven in [88], for any pullbacks of the diffeomorphism of unity form $p(x) \,=\, x \,+\cdots$

Remark: Seeking for Calabi-Yau ODEs, Almkvist et al. have obtained [91] a quite large list of fourth order ODEs, which are MUM by definition and have, by construction, the *integrality* for the solution-series‡ analytic at $x \,=\, 0$. Looking at the Yukawa coupling of these ODEs is a way to define *equivalence classes up to pullbacks* of ODEs sharing the same Yukawa coupling. This "wraps in the same bag" all the linear ODEs that are the same *up to pullbacks*. Let us recall how difficult it is to see if a given Calabi-Yau ODE has, up to operator equivalence, and up to pullback, a hypergeometric function solution [32, 33], because finding the pullback is extremely difficult [32, 33]. We may have, for the Ising model, some ${}_{n+1}F_n$ hypergeometric function prejudice [32, 33]: it is, then, important to have an invariant that is independent of this pullback (we cannot find most of the time).

Remark: The Yukawa coupling is not preserved by the operator equivalence. Two linear differential operators, that are homomorphic, do not necessarily have the same Yukawa coupling (see Appendix J).

8.2. Yukawa couplings in terms of determinants

Another way to understand this fundamental *pullback invariance*, amounts to rewriting the Yukawa coupling [122, 88], not from the definition usually given in the literature (second derivative with respect to the ratio of periods), but in terms of determinants of solutions (Wronskians, ...) that naturally present nice covariance properties with respect to pullback transformations (see Appendix J).

We have the alternative definition for the *Yukawa coupling* given in Appendix J:

$$K(q) \;=\; \Big(q\cdot\frac{d}{dq}\Big)^2\Big(\frac{y_2}{y_0}\Big) \;=\; \frac{W_1^3\cdot W_3}{W_2^3}, \qquad (111)$$

where the determinantal variables W_m's are the determinants built from the four solutions of the MUM differential operator. This alternative definition, in terms of these W_m's, enables to understand the *remarkable invariance of the Yukawa coupling by pullback transformations* [33]. These determinantal variables W_m quite naturally, and canonically, yield to introduce another "Yukawa coupling" (which, in fact, *corresponds to the Yukawa coupling of the adjoint operator* (see J.12)). This "adjoint Yukawa coupling" is *also invariant by pullbacks*. It has, for the previous example, the following series expansion with integer coefficients:

$$K^\star(q) \;=\; 1 \;+32\cdot q \;+4896\cdot q^2 \;+702464\cdot q^3 \;+\;\cdots \qquad (112)$$

‡ Hence, these operators are G-operators. One can also, calculating their p-curvatures, check, directly, that the Calabi-Yau operators in Almkvist et al. [91] tables are actually globally nilpotent [28], thus yielding automatically the rationality of the exponents for all the singularities

which actually identifies with (110). The equality of the Yukawa coupling for this order-four operator, and for its (formal) adjoint operator, is a straightforward consequence of the fact that the order-four operator annihilating ${}_4F_3\Big([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}], [1, 1, 1], 256\,x\Big)$ is exactly *self-adjoint*, and, more generally, of the fact that the order-four operator, annihilating (107), is conjugated to its adjoint by a simple function (which is nothing but the denominator of the pullback).

8.3. Modularity

This example, with its corresponding relations (107), (108), (110), (112) may suggest a quite wrong prejudice that the *integrality of the solution* of an order-four linear differential operator automatically yields to the integrality of the nome, mirror map and Yukawa coupling, that we will call, for short, "*modularity*". This is *far from being the case*, as can be seen, for instance, in the following interesting example, where the nome and Yukawa coupling $K(q)$ *do not correspond to globally bounded series*, when the ${}_4F_3$ solution of the order-four operator as well as the Yukawa coupling *seen as a function of* x, $K(x)$, are, actually, both *series with integer coefficients*.

Let us consider the following ${}_4F_3$ hypergeometric function which is clearly a Hadamard product of algebraic functions and, thus, the diagonal of a rational function:

$$
\begin{aligned}
&{}_4F_3\Big([\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{4}], [1, 1, 1], x\Big) \\
&\qquad = \quad (1-x)^{-1/3} \star (1-x)^{-1/2} \star (1-x)^{-1/4} \star (1-x)^{-3/4} \\
&\qquad = \quad \mathsf{Diag}\Big((1-z_1)^{-1/3}\,(1-z_2)^{-1/2}\,(1-z_3)^{-1/4}\,(1-z_4)^{-3/4}\Big),
\end{aligned}
$$

It is therefore globally bounded:

$$
\begin{aligned}
{}_4F_3([\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{4}], [1, 1, 1], 2304\,x) \quad = \quad & 1 \; + 72\,x \; + 45360\,x^2 \; + 46569600\,x^3 \\
& + 59594535000\,x^4 \; + 86482063571904\,x^5 \; + 136141986298526208\,x^6 \\
& + 226888189910421811200\,x^7 \; + 394399917777684601926000\,x^8 \\
& + 708188604075430924446000000\,x^9 \quad + \cdots
\end{aligned}
\tag{113}
$$

Its Yukawa coupling, seen as a function of x, is actually a *series with integer coefficient* in x:

$$
\begin{aligned}
K(x) \; = \; & 1 \; + 480\,x \; + 872496\,x^2 \; + 1728211968\,x^3 \; + 3566216754432\,x^4 \\
& + 7536580798814208\,x^5 \; + 16177041308360579328\,x^6 \\
& + 35105183794659521064960\,x^7 + 76799014669577085362391024\,x^8 \\
& + 169059790576811511759706311168\,x^9 \quad + \cdots
\end{aligned}
\tag{114}
$$

However, do note that the series, in term of the nome, is *not globally bounded*:

$$
\begin{aligned}
K(q) \; = \; & 1 \; + 480\;q \; + 653616\;q^2 \; + 942915456\;q^3 \; + 1408019875200\;q^4 \\
& + 2146833138536640\;q^5 \quad + \; \cdots \\
& + \; 571436303929319146711343817202689132288\;\frac{q^{12}}{11} \quad + \; \cdots
\end{aligned}
\tag{115}
$$

In fact, the nome $q(x)$, and the mirror map $x(q)$, are *also not globally bounded*. Note that in this example, the non integrality appears at order twelve (for $x(q)$, $q(x)$ and $K(q)$). If the prime 11 in the denominator in (115) was the only one, one could recast the series into a series with integer coefficients introducing another rescaling $2304\,x \rightarrow 11 \times 2304\,x$. But, in fact, we do see the appearance of an *infinite number of other primes* at higher orders denominators in $x(q)$, $q(x)$ and $K(q)$.

8.4. Hadamard products of ω_n's

Let us consider the two order-two operators

$$\omega_2 \;=\; D_x^2 \;+\; \frac{(96\,x+1)}{(64\,x+1)\cdot\, x}\cdot D_x \;+\; \frac{4}{(64\,x+1)\,x}, \tag{116}$$

$$\omega_3 \;=\; D_x^2 \;+\; \frac{(45\,x+1)}{(27\,x+1)\cdot\, x}\cdot D_x \;+\; \frac{3}{(27\,x+1)\,x}, \tag{117}$$

which are associated with two modular forms corresponding, on their associated nomes q, to the transformations $q \rightarrow q^2$ and $q \rightarrow q^3$ respectively (multiplication of τ, the ratio of their periods by 2 and 3), as can be seen on their respective solutions:

$$\begin{aligned} {}_2F_1\Big([\frac{1}{4}, \frac{1}{4}], [1], -64\,x\Big) \;&=\; (1\,+256\,x)^{-1/4}\cdot\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728\,x}{(1\,+256\,x)^3}\Big) \\ &=\; (1\,+16\,x)^{-1/4}\cdot\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728\,x^2}{(1\,+16\,x)^3}\Big) \\ =\; 1\; -4\,x\,+100\,x^2\,-3600\,x^3\,&+152100\,x^4\,-7033104\,x^5\,+344622096\,x^6 \\ &-17582760000\,x^7\,+924193822500\,x^8\,-49701090010000\,x^9\;+\;\cdots \end{aligned} \tag{118}$$

$$\begin{aligned} &\Big((1\,+27\,x)\,(1\,+243\,x)^3\Big)^{-1/12}\cdot\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728\,x}{(1\,+243\,x)^3\,(1\,+27\,x)}\Big) \\ &=\; \Big((1\,+27\,x)\,(1\,+3\,x)^3\Big)^{-1/12}\cdot\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728\,x^3}{(1\,+3\,x)^3\,(1\,+27\,x)}\Big) \\ &=\; {}_2F_1\Big([\frac{1}{3}, \frac{1}{3}], [1], -27\,x\Big) \;=\; 1\;-3\,x\,+36\,x^2\,-588\,x^3\,+11025\,x^4\,-223587\,x^5 \\ &\qquad +4769856\,x^6\,-105423552\,x^7\,+2391796836\,x^8\,-55365667500\,x^9\;+\;\cdots \end{aligned} \tag{119}$$

The relation between the two Hauptmodul pullbacks in (118)

$$u \;=\; \frac{1728\,x}{(1\,+256\,x)^3}, \qquad\qquad v \;=\; \frac{1728\,x^2}{(1\,+16\,x)^3}, \tag{120}$$

corresponds to the (genus-zero) fundamental modular curve:

$$\begin{aligned} 1953125\,u^3v^3\;-187500\,u^2v^2\cdot\,(v+u)\;+375\,uv\cdot\,\big(16\,u^2+16\,v^2\,-4027\,uv\big) \\ -\,64\,(u\,+v)\cdot\,\big(v^2+1487\,uv+u^2\big)\;+110592\,uv \;=\; 0, \end{aligned} \tag{121}$$

The relation between the two Hauptmodul pullbacks in (119)

$$u \;=\; \frac{1728\,x}{(1\,+243\,x)^3\,(1\,+27\,x)}, \qquad v \;=\; \frac{1728\,x^3}{(1\,+3\,x)^3\,(1\,+27\,x)}, \tag{122}$$

corresponds to the (genus-zero) modular curve:

$$\begin{aligned} 262144000000000\,u^3v^3\cdot\,(u\,+v)\;+4096000000\,u^2v^2\cdot\,(27\,v^2+27\,u^2-45946\,uv) \\ +\,15552000\,uv\cdot\,(u\,+v)\cdot\,\big(v^2+241433\,uv+u^2\big) \\ +\,729\,(u^4\,+v^4)\;-779997924\,(u^3v+\,uv^3)\;+1886592284694\,u^2v^2 \\ +\,2811677184\,uv\cdot\,(u\,+v)\;-2176782336\,uv \;=\; 0. \end{aligned} \tag{123}$$

Similarly, one can consider the order-two operators ω_n associated with other modular forms corresponding to $\tau \rightarrow n \cdot \tau$. The ω_n's can be simply deduced from Maier [123], for modular forms corresponding to genus-zero curves i.e. for $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25$. After a simple rescaling, one gets series with integer coefficients. For instance considering the linear differential operator $\mathcal{L}_7$ (annihilating the modular form h_7) in Table 13 of [116]

$$\mathcal{L}_7 \quad = \quad D_x^2 \; + \; \frac{7\,x^2+65\,x+147}{3\,(x^2+13\,x+49)\,x} \cdot D_x \; + \; \frac{4\,x+21}{9\,(x^2+13\,x+49)\, \cdot \, x}, \qquad (124)$$

one has the modular form solution

$$\begin{aligned} & D_7(x)^{1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1]; \frac{1728}{j_7(x)}\Big) \\ & \quad = \quad \frac{7^{7/6}}{x^{2/3}} \cdot D_7'(x)^{1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1]; \frac{1728}{j_7'(x)}\Big) \qquad (125) \\ & \quad = \quad 1 \; - \frac{1}{21}\,x \; + \frac{11}{3087}\,x^2 \; - \frac{380}{1361367}\,x^3 \; + \frac{3887}{200120949}\,x^4 \quad + \cdots \end{aligned}$$

where

$$\begin{aligned} j_7(x) \; &= \; \frac{\big(x^2+13\,x+49\big)\,\big(x^2+5\,x+1\big)^3}{x}, \qquad & j_7'(x) \; &= \; j_7\Big(\frac{49}{x}\Big), \\ D_7(x) \; &= \; \frac{49}{(x^2+13\,x+49)\cdot(x^2+5\,x+1)^3}, \qquad & D_7'(x) \; &= \; D_7\Big(\frac{49}{x}\Big), \qquad (126) \end{aligned}$$

The series (125) is globally bounded. Rescaling the x into $3^2\, 7^2 \cdot x \, = \, 441 \cdot x$, Maier's linear differential operator (124) becomes

$$\omega_7 \;\; = \;\; D_x^2 \; + \; \frac{1\,+195\,x+\,9261\,x^2}{(1\,+117\,x+\,3969\,x^2)\cdot\,x} \cdot D_x \; + \; \frac{21\cdot\,(1\,+\,84\,x)}{(1\,+117\,x+\,3969\,x^2)\cdot\,x}, \quad (127)$$

which has the following series with *integer* coefficients:

$$\begin{aligned} & D_7(441\,x)^{1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1]; \frac{1728}{j_7(441\,x)}\Big) \qquad (128) \\ & \quad = \quad 1 \; - 21\,x \; + 693\,x^2 \; - 23940\,x^3 \; + 734643\,x^4 \; - 13697019\,x^5 \\ & \qquad \qquad - 494620749\,x^6 \; + 83079255420\,x^7 \; - 6814815765975\,x^8 \quad + \; \cdots \end{aligned}$$

The two operators ω_2 and ω_3 have a "modularity" property: their series expansions analytic at $x = 0$, (118) and (119), *as well as* the corresponding nomes, mirror maps are series with integer coefficients. The Hadamard product is a quite natural transformation to introduce because *it preserves the global nilpotence of the operators*, *it preserves the integrality of series-solutions*, and it is a *natural transformation to introduce when seeking for diagonals of rational functions*¶. Let us perform the Hadamard product of these two operators. With some abuse of language [33], the Hadamard product of the two order-two operators (116) and (117)

$$\begin{aligned} H_{2,3} \;\; = \;\; & D_x^4 \;\; + 6\,\frac{(2064\,x-1)}{(1728\,x-1)\cdot\,x} \cdot D_x^3 \;\; + \frac{(19020\,x-7)}{(1728\,x-1)\cdot\,x^2} \cdot D_x^2 \\ & + \frac{(4788\,x-1)}{(1728\,x-1)\cdot\,x^3} \cdot D_x \;\; + \frac{12}{(1728\,x-1)\cdot\,x^3}, \qquad (129) \end{aligned}$$

¶ And, consequently, has been heavily used to build Calabi-Yau-like ODEs (see Almkvist et al. [88]).

is defined as the (minimal order) linear differential operator having, as a solution, the Hadamard product of the solution-series (118) and (119), which is, by construction, a series with integer coefficients. This series is, of course, nothing but the expansion of the hypergeometric function:

$$
{}_4F_3([\frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{3}], [1, 1, 1], 1728\,x) \tag{130}
$$
$$
= \quad {}_2F_1([\frac{1}{4}, \frac{1}{4}], [1], -64\,x) \star {}_2F_1([\frac{1}{3}, \frac{1}{3}], [1], -27\,x).
$$

The Hadamard product of the order-two operator (116) with itself (Hadamard square)

$$
H_{2,2} \quad = \quad D_x^4 \quad + 2\,\frac{(14336\,x-3)}{(4096\,x-1)\cdot\, x} \cdot D_x^3 \quad + \frac{(42496\,x-7)}{(4096\,x-1)\cdot\, x^2} \cdot D_x^2
$$
$$
+ \frac{(9984\,x-1)}{(4096\,x-1)\cdot\, x^3} \cdot D_x \quad + \frac{16}{(4096\,x-1)\cdot\, x^3}, \tag{131}
$$

is defined as the (minimal order) linear differential operator having the series-solution

$$
1 \; + 16\,x \; + 10000\,x^2 \; + 12960000\,x^3 \; + 23134410000\,x^4 \; + 49464551874816\,x^5 \;\; + \; \cdots \, ,
$$

which is the Hadamard product of the solution-series (118) with itself. This series is, of course, nothing but the expansion of:

$$
{}_4F_3([\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}], [1, 1, 1], 4096\,x) \tag{132}
$$
$$
= \quad {}_2F_1([\frac{1}{4}, \frac{1}{4}], [1], -64\,x) \star {}_2F_1([\frac{1}{4}, \frac{1}{4}], [1], -64\,x).
$$

This operator $H_{2,2}$ is a MUM operator. We can, therefore, define, without any ambiguity, the nome (and mirror map) and Yukawa coupling of this order-four operator [33]. One finds out that the nome†, and the mirror map (and the Yukawa coupling as a function of the x variable), are *not globally bounded*: they *cannot* be reduced, by one rescaling, to series with integer coefficients.

Similarly, one can also introduce the Hadamard square of (117)

$$
H_{3,3} \quad = \quad D_x^4 \quad + 6\,\frac{(891\,x-1)^3}{(729\,x-1)\cdot\, x} \cdot D_x^3 \quad + 7\,\frac{(1215\,x-1)}{(729\,x-1)\cdot\, x^2} \cdot D_x^2
$$
$$
+ \frac{(2295\,x-1)}{(729\,x-1)\cdot\, x^3} \cdot D_x \quad + \frac{9}{(729\,x-1)\cdot\, x^3}, \tag{133}
$$

which has the hypergeometric solution:

$$
{}_4F_3([\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}], [1, 1, 1], 729\,x) \tag{134}
$$
$$
= \quad {}_2F_1([\frac{1}{3}, \frac{1}{3}], [1], -27\,x) \star {}_2F_1([\frac{1}{3}, \frac{1}{3}], [1], -27\,x).
$$

Let us remark that the three linear differential operators (129), (131) and (133), are MUM and of order four. However, they are not of the Calabi-Yau type.

† The nome of the Hadamard product of two operators has no simple relation with the nome of these two linear differential operators.

8.5. Hadamard products versus Calabi-Yau ODEs

This is not the case for other values of n and m. For instance one can introduce‡ $H_{4,4} = \omega_4 \star \omega_4$, the Hadamard square of ω_4, which is an irreducible order-four linear differential operator, and has the hypergeometric solution already encountered for some n-fold integrals of the decomposition of the full magnetic susceptibility of the Ising model [32, 33]:

$$ {}_4F_3([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}], [1, 1, 1], 256\, x) \qquad (135)$$
$$ = \quad {}_2F_1([\frac{1}{2}, \frac{1}{2}], [1], -16\, x) \star {}_2F_1([\frac{1}{2}, \frac{1}{2}], [1], -16\, x). $$

The associated operator having (135) as a solution, obeys the "Calabi-Yau condition" that its exterior square is of *order five*.

Let us give in a table the orders (which go from 4 to 20) of the various $H_{m,n} = H_{n,m}$ Hadamard products of the order-two operators associated with the (genus-zero) modular forms operators ω_n and ω_m:

n\m	2	3	4	5	6	7	8	9	10	12	13	16	18	25
2	4	4	4	6	4	6	4	4	10	8	10	8	12	14
3		4	4	6	4	6	4	4	10	8	10	8	12	14
4			4 $*$	6	4 $*$	6	4 $*$	4 $*$	10	8	10	8	12	14
5				6	6	8	6	6	12	10	12	10	14	16
6					4 $*$	6	4 $*$	4 $*$	10	8	10	8	12	14
7						6	6	6	12	10	12	10	14	16
8							4 $*$	4 $*$	10	8	10	8	12	14
9								4 $*$	10	8	10	8	12	14
10									10	14	16	14	18	20
12										8	14	12	16	18
13											10	14	18	20
16												8	16	18
18													12	20
25														14

where the star $*$ denotes Calabi-Yau ODEs♯.

The following operators are of order four: $H_{2,2}$, $H_{2,3}$, $H_{2,4}$, $H_{2,6}$, $H_{2,8}$, $H_{2,9}$, $H_{3,3}$, $H_{3,4}$, $H_{3,6}$, $H_{3,8}$, $H_{3,9}$, ... Their exterior squares, which are of order six, do not have rational solutions¶.

The following operators are of order six: $H_{2,5}$, $H_{2,7}$, $H_{3,5}$, $H_{3,7}$, $H_{4,5}$, $H_{4,7}$, $H_{5,5}$, $H_{5,6}$, $H_{5,8}$, $H_{5,9}$, $H_{6,7}$, $H_{7,7}$, $H_{7,8}, H_{7,9}$, ... Their exterior square, which are of order fifteen, do not have rational solutions (and cannot be homomorphic to higher order Calabi-Yau linear ODEs).

Remarkably the following ten order-four operators $H_{4,4}$, $H_{4,6}$, $H_{4,8}$, $H_{4,9}$, $H_{6,6}$, $H_{6,8}$, $H_{6,9}$, $H_{8,8}$, $H_{8,9}$, $H_{9,9}$ (with a star in the previous table) are all MUM, and

‡ To get the Hadamard product of two linear differential operators use, for instance, Maple's command gfun[hadamardproduct].

♯ Recall that Calabi-Yau ODEs are defined by a list of constraints [88], the most important ones being, besides being MUM, that their *exterior square are of order five*. There are more exotic conditions like the cyclotomic condition on the monodromy at ∞, see Proposition 3 in [91].

¶ They cannot be homomorphic to Calabi-Yau ODEs.

are such that their exterior squares are of order five§: they are *Calabi-Yau ODEs*. Actually the nome, mirror map and Yukawa coupling series are *series with integer coefficients for all these order four Calabi-Yau operators*. The Yukawa coupling series of these Calabi-Yau operators are respectively, for $H_{4,4}$

$$K(q) \; = \; K^{\star}(q) \; = \; 1 \; + 32 \cdot q \; + 4896 \cdot q^2 \; + 702464 \cdot q^3 \; + 102820640 \cdot q^4 \; + 15296748032 \cdot q^5 \; + 2302235670528 \cdot q^6 \; + 349438855544832 \cdot q^7 \; + 53378019187206944 \cdot q^8 \; + 8194222260681725696 \cdot q^9 \; + \; \cdots, \qquad (136)$$

which is $Number 3$ in Almkvist et al. large tables of Calabi-Yau ODEs [91], and is the well-known one for†† ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1], 256\, x)$, for $H_{4,6}$:

$$K(q) \; = \; K^{\star}(q) \; = \; 1 \; + 20 \cdot q \; + 36 \cdot q^2 \; + 15176 \cdot q^3 \; + 486564 \cdot q^4 \; + 21684020 \cdot q^5 \; + 1209684456 \cdot q^6 \; + 58513394904 \cdot q^7 \; + 2921860726948 \cdot q^8 \; + 141376772107064 \cdot q^9 \; + \; \cdots \qquad (137)$$

which is $Number 137$ in tables [91].

We give, in Appendix L, the expansion of the Yukawa coupling for a set of other $H_{m,n}$ that are Calabi-Yau: in particular their exterior square is order *five* (not six as one could expect for a generic irreducible order-four operator). It will be shown, in a forthcoming publication, that the fact that the order five exterior power property occurs means that these operators are necessarily *conjugated* (by an algebraic function) to their adjoints. Thus, the "adjoint Yukawa coupling" $K^{\star}(q)$ is necessarily equal to the Yukawa coupling $K(q)$ for these operators.

Remark: The operator having the Hadamard product of the two HeunG functions $HeunG(a, q, 1, 1, 1, 1; x)$ and $HeunG(A, Q, 1, 1, 1, 1; x)$ as a solution reads:

$$(x-1)\,(x-a)\,(x-A)\,(x-A\,a)\,(A\,a-x^2)^2 \cdot x^3 \cdot D_x^4 \qquad (138)$$
$$+ 2\,(x^2 - A\,a) \cdot U_3 \cdot x^2 \cdot D_x^3 \; - U_2 \cdot x \cdot D_x^2 \; - U_1 \cdot D_x \; + U_0,$$

where the polynomials U_n are given in Appendix K.

The exterior square of this order-four operator (138) is of order *five* for *any value* of the parameters a, q, A, Q (instead of the order-six one expects for the exterior square of a generic irreducible order-four operator).

The HeunG functions solutions of the form $HeunG(a, q, 1, 1, 1, 1; x)$ are an interesting set of HeunG functions. They verify the following (six Möbius) identities [124]:

$$HeunG(a, q, 1, 1, 1, 1;\; x) \; = \; HeunG\Big(\frac{1}{a}, \frac{q}{a}, 1, 1, 1, 1;\; \frac{x}{a}\Big)$$
$$= \; \frac{1}{1-x} \cdot HeunG\Big(\frac{a}{a-1}, \frac{a-q}{a-1}, 1, 1, 1, 1;\; -\frac{x}{1-x}\Big)$$
$$= \; \frac{1}{1-x/a} \cdot HeunG\Big(\frac{1}{1-a}, \frac{q-1}{a-1}, 1, 1, 1, 1;\; -\frac{x}{a-x}\Big) \qquad (139)$$
$$= \; \frac{a}{a-x} \cdot HeunG\Big(1-a,\; 1-q, 1, 1, 1, 1;\; \frac{(a-1)\cdot x}{a-x}\Big)$$
$$= \; \frac{1}{1-x} \cdot HeunG\Big(\frac{a-1}{a}, \frac{a-q}{a}, 1, 1, 1, 1;\; -\frac{x}{1-x} \cdot \frac{a-1}{a}\Big).$$

§ They are conjugated to their (formal) adjoint by a function.
†† Actually $H_{4,4}$ is exactly the operator for ${}_4F_3([1/2, 1/2, 1/2, 1/2], [1, 1, 1], 256\, x)$.

The ten linear differential operators denoted by a star $*$ in the previous table are all of this form: they have the Hadamard product of two HeunG functions solutions of the form $HeunG(a, q, 1, 1, 1, 1; x)$ as a solution. Note, however, that this HeunG-viewpoint of the most interesting $H_{m,n}$'s does not really help. Even inside this restricted set of HeunG functions solutions of the form $HeunG(a, q, 1, 1, 1, 1; x)$ it is hard to find exhaustively the values of the two parameters a and of the accessory parameter† q such the series $HeunG(a, q, 1, 1, 1, 1; x)$ is globally bounded‡, or, just, such that the order-two operator having $HeunG(a, q, 1, 1, 1, 1; x)$ as a solution is globally nilpotent (see Appendix K).

The order-four operators $H_{3,3}$, $H_{3,4}$, are all MUM operators¶, but, similarly to the situation encountered with $H_{2,2}$, their nome, mirror map and Yukawa couplings are *not globally bounded*.

Many $H_{m,n}$ are not MUM, for instance the order-eight operator $H_{12,12}$, or the order-six operator $H_{3,7}$, are *not MUM*. Concerning $H_{3,7}$ and as far as its six solutions are concerned, it is "like" the four solutions of an order-four MUM operator, together with the two solutions of another order-two MUM operator, but the order-six operator $H_{3,7}$ is not a direct-sum of an order-four and order-two operator. We have two solutions analytical at $x = 0$ (no $\ln(x)$), two solutions with a $\ln(x)$. A linear combination of these two solutions analytical at $x = 0$ is, by construction a series with *integer coefficients* (the Hadamard product of the two series with integer coefficients which are the initial ingredients in this calculation), when the other linear combinations are *not globally bounded*.

8.6. "Atkin" transformations

It is worth noting that the globally bounded character of some of these $H_{m,n}$, at $x = 0$, is simply related to the globally bounded character at $x = \infty$ of a conjugate operator, as a consequence of some simple homomorphism relation with their transformed by "Atkin" pullbacks:

$$\begin{aligned} x^{1/4} \cdot H_{2,2}(x) &= H_{2,2}\Big(x \rightarrow \frac{1}{2^{24}\, x}\Big) \cdot x^{1/4}, \\ x^{1/3} \cdot H_{3,3}(x) &= H_{3,3}\Big(x \rightarrow \frac{1}{3^{12}\, x}\Big) \cdot x^{1/3}, \end{aligned} \tag{140}$$

$$x^{1/2} \cdot H_{4,4}(x) = H_{4,4}\Big(x \rightarrow \frac{1}{2^{16}\, x}\Big) \cdot x^{1/2}, \quad x \cdot H_{6,6}(x) = H_{6,6}\Big(x \rightarrow \frac{1}{2^{6}\, 3^{4}\, x}\Big) \cdot x,$$

$$x \cdot H_{8,8}(x) = H_{8,8}\Big(x \rightarrow \frac{1}{2^{10}\, x}\Big) \cdot x, \qquad x \cdot H_{9,9}(x) = H_{9,9}\Big(x \rightarrow \frac{1}{3^{6}\, x}\Big) \cdot x,$$

but also for the order-six operators

$$x^{1/2} \cdot H_{5,5}(x) = H_{5,5}\Big(x \rightarrow \frac{1}{2^{12}\, 5^{6}\, x}\Big) \cdot x^{1/2}, \tag{141}$$

$$x^{2/3} \cdot H_{7,7}(x) = H_{7,7}\Big(x \rightarrow \frac{1}{3^{8}\, 7^{4}\, x}\Big) \cdot x^{2/3}, \tag{142}$$

† The accessory parameter appears in many applications as a spectral parameter [124].
‡ Along this line see the paper by Zagier [96] on integral solutions of Apéry-like equations.
¶ Note that the Hadamard product of two MUM ODEs is not necessarily a MUM ODE: the order-six operator $H_{3,7}$ is *not* MUM.

and the order-eight operators

$$x^2 \cdot H_{12,12}(x) \quad = \quad H_{12,12}\Big(x \rightarrow \frac{1}{2^4\, 3^2\, x}\Big) \cdot x^2, \qquad (143)$$

$$x^2 \cdot H_{16,16}(x) \quad = \quad H_{16,16}\Big(x \rightarrow \frac{1}{2^6\, x}\Big) \cdot x^2, \qquad (144)$$

or the order-ten operators

$$x^{3/2} \cdot H_{10,10}(x) \quad = \quad H_{10,10}\Big(x \rightarrow \frac{1}{2^8\, 5^2\, x}\Big) \cdot x^{3/2}, \qquad (145)$$

$$x^{7/6} \cdot H_{13,13}(x) \quad = \quad H_{13,13}\Big(x \rightarrow \frac{1}{2^8\, 3^3\, 13^2\, x}\Big) \cdot x^{7/6}, \qquad (146)$$

or the order-twelve operator

$$x^3 \cdot H_{18,18}(x) \quad = \quad H_{18,18}\Big(x \rightarrow \frac{1}{2^2\, 3^2\, x}\Big) \cdot x^3, \qquad (147)$$

or the order-fourteen operator

$$x^{5/2} \cdot H_{25,25}(x) \quad = \quad H_{25,25}\Big(x \rightarrow \frac{1}{2^8\, 5^2\, x}\Big) \cdot x^{5/2}. \qquad (148)$$

to be compared with

$$x^{1/4} \cdot \omega_2(x) \; = \; \omega_2\Big(x \rightarrow \frac{1}{2^{12}\, x}\Big) \cdot x^{1/4}, \qquad x^{1/3} \cdot \omega_3(x) \; = \; \omega_3\Big(x \rightarrow \frac{1}{3^6\, x}\Big) \cdot x^{1/3},$$

$$x^{1/2} \cdot \omega_4(x) \; = \; \omega_4\Big(x \rightarrow \frac{1}{2^8\, x}\Big) \cdot x^{1/2}, \qquad x \cdot \omega_6(x) \; = \; \omega_6\Big(x \rightarrow \frac{1}{2^3\, 3^2\, x}\Big) \cdot x,$$

$$x \cdot \omega_8(x) \; = \; \omega_8\Big(x \rightarrow \frac{1}{2^5\, x}\Big) \cdot x, \qquad x \cdot \omega_9(x) \; = \; \omega_9\Big(x \rightarrow \frac{1}{3^3\, x}\Big) \cdot x,$$

and:

$$x^{1/2} \cdot \omega_5(x) \quad = \quad \omega_5\Big(x \rightarrow \frac{1}{2^6\, 5^3\, x}\Big) \cdot x^{1/2}, \qquad (149)$$

$$x^{2/3} \cdot \omega_7(x) \quad = \quad \omega_7\Big(x \rightarrow \frac{1}{3^4\, 7^2\, x}\Big) \cdot x^{2/3}, \qquad \cdots$$

Note, however, that these properties (140) are no longer valid for the off-diagonal operators $H_{m,n}$, $m \neq n$. For instance, the other order-four Calabi-Yau operators $H_{4,6}$, $H_{4,8}$, $H_{4,9}$ are *not globally bounded* at $x = \infty$: if one changes these operators by a $x \rightarrow 1/x$ pullback, the new order-four operators after a well-suited conjugation by x^r (r rational number) are *not globally bounded*. For instance $H_{4,6}(x \rightarrow 1/x)$ requires to be conjugated by $x^{1/2}$ (Puiseux series): after conjugation by $x^{1/2}$, the series analytical at $x = 0$ is not globally bounded.

Remark: Let us denote A_n the constant in the "Atkin" involution $x \rightarrow A_n/x$ for modular form of order n (i.e. $\tau \rightarrow n\,\tau$). We denote by $\omega_n(x)$, the order-two operator associated with a modular form of order n.

One has

$$x^{r_n} \cdot \omega_n(x) \quad = \quad \omega_n\Big(x \rightarrow \frac{1}{A_n\, B_n^2\, x}\Big) \cdot x^{r_n}, \qquad (150)$$

where the B_n's are integers, and, quite remarkably¶, one also has for $H_{n,n}(x)$ the

¶ If one forgets, for a second, that sum and multiplication do not commute, it is tempting to see naively these conjugation results (151), on the $H_{n,n}$'s, as a straight consequence of the relation, like (150), on the ω_n's, since the $H_{n,n}$'s are Hadamard squares of the ω_n's, the constant in the pullback involution in the right-hand side of (151) being the square of the constant in (150). This is not the case.

Hadamard product of two $\omega_n(x)$'s

$$x^{r_n} \cdot H_{n,n}(x) \quad = \quad H_{n,n}\Big(x \rightarrow \frac{1}{A_n^2\, B_n^4\, x}\Big) \cdot x^{r_n}. \qquad (151)$$

The values of A_n, B_n and r_n are given in the following table:

n	2	3	4	5	6	7	8	9	10	12	13	16	18	25
A_n	2^{12}	3^6	2^8	5^3	$2^3\,3^2$	7^2	2^5	3^3	$2^2\,5$	$2^2\,3$	13	2^3	$2\cdot 3$	5
B_n	1	1	1	2^2	1	3^2	1	2	1	1	$2^2\,3^2$	1	1	2^2
r_n	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{2}{3}$	1	1	$\frac{3}{2}$	2	$\frac{7}{6}$	2	3	$\frac{5}{2}$

8.7. ${}_2F_1([1/N,\, 1/N],\, [1],\, x)$ *hypergeometric functions*

Modular forms can always be written as the hypergeometric function ${}_2F_1$, up to an algebraic pre-factor, and *up to a pullback* (see Maier [123]). Relations (130), (132), (134) and (135) underline the special role of modular forms that can be written as ${}_2F_1$ with no algebraic pre-factor, and no pullbacks (see Table 15 in [123]), namely ${}_2F_1([1/4,\, 1/4],\, [1],\, -64\,x)$, ${}_2F_1([1/3,\, 1/3],\, [1],\, -27\,x)$, ${}_2F_1([1/2,\, 1/2],\, [1],\, -16\,x)$. Along this line it is worth considering the hypergeometric functions ${}_2F_1([1/N,\, 1/N],\, [1],\, -N^3\,x)$ which always yield globally bounded series, together with their associated linear differential operators. The nome for these operators *does not* (generically) correspond to globally bounded series. One notes, however, that $N\, = \,6$ is "special", yielding series with integer coefficients for the hypergeometric function†

$$\begin{aligned} &{}_2F_1\Big([\frac{1}{6},\, \frac{1}{6}],\, [1],\, -432\,x\Big) \qquad (152)\\ &\quad = \quad (1\, +\, 432\,x)^{-1/6} \cdot {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, \frac{1728\,x}{(1\, +\, 432\,x)^2}\Big) \\ &\quad = \quad 1\, -12\,x\, +1764\,x^2\, -397488\,x^3\, +107619876\,x^4\, -32285962800\,x^5 \\ &\qquad\qquad +10342270083600\,x^6\, -3467404345579200\,x^7\, + \,\cdots \end{aligned}$$

but also for the corresponding nome

$$\begin{aligned} q \quad = \quad & x\, -120\,x^2\, +24660\,x^3\, -6322720\,x^4\, +1828573410\,x^5\, -570359919024\,x^6 \\ & +187363061411720\,x^7\, -63912709875600960\,x^8\, + \,\cdots \qquad (153) \end{aligned}$$

as well as the mirror map:

$$\begin{aligned} x(q) \quad = \quad & q\, +120\,q^2\, +4140\,q^3\, +166720\,q^4\, -6012210\,q^5\, +1165528224\,q^6 \\ & -178811454280\,q^7\, +29512658112000\,q^8\, + \,\cdots \qquad (154) \end{aligned}$$

In fact the hypergeometric function ${}_2F_1([1/6,\, 1/6],\, [1],\, x)$ (as well as the hypergeometric function ${}_2F_1([1/6,\, 5/6],\, [1],\, 1-x))$ is actually "special" (as was first seen by Ramanujan, see for instance Cooper [125]). In Appendix M the modular form character of (152) is made crystal clear in a quite heuristic way.

Remark: Using the ${}_2F_1([a, b], [1], x)$ hypergeometric functions associated to modular forms‡ (namely $[a, b]\, = \, [1/2, 1/2]$, $[1/3, 1/3]$, $[1/4, 1/4]$, $[1/6, 1/6]$,

† We use identity ${}_2F_1([1/6, 1/6], [1], x)\, = \,(1-x)^{-1/6} \cdot {}_2F_1([1/12, 5/12], [1], -4\,x/(1-x)^2)$, which singles out the known [113] pullback $x \, \rightarrow \, -4\,x/(1-x)^2$.

‡ For $[1/2, 1/2]$, $[1/3, 1/3]$, $[1/4, 1/4]$, $[1/6, 1/6]$, see the Ramanujan's theories to alternative basis [125] for other "signatures". For $[1/8, 3/8]$ see the third example in Appendix A.

[1/3, 2/3], [1/3, 1/6], [1/6, 5/6], [1/4, 3/4], [1/8, 3/8], [1/12, 5/12], ...), one can build $_4F_3$ globally bounded examples by simple Hadamard products of these selected $_2F_1$. We give in Appendix P a miscellaneous set of identities expressing HeunG functions, or modular forms, as the previously selected $_2F_1$ hypergeometric functions with *two pullbacks*.

8.8. Modularity and Hypergeometric series with coefficients ratio of factorials

As a consequence of the classification by Beukers and Heckman [111] of all algebraic $_nF_{n-1}$'s, the $_8F_7$ hypergeometric series

$$ _8F_7\Big(\Big[\frac{1}{30}, \frac{7}{30}, \frac{11}{30}, \frac{13}{30}, \frac{17}{30}, \frac{19}{30}, \frac{23}{30}, \frac{29}{30}\Big], \Big[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}\Big], \; 2^{14}\, 3^9\, 5^5\, x\Big), $$

has *integer coefficients*, and is an *algebraic function*. The Galois group belonging to this function is the Weyl group $W(E_8)$ which has 696729600 elements [126]. It is an *algebraic series* of degree 483840. More precisely, it was noticed [127] by Rodriguez-Villegas that the previous power series reads:

$$ \sum_{n=0}^{\infty} \frac{(30\,n)!\; n!}{(15\,n)!\;(10\,n)!\;(6\,n)!} \cdot x^n, \tag{155} $$

which is precisely the series introduced by Chebyshev during his work [128] on the distribution of prime numbers to establish the estimate

$$ 0.92\;\frac{x}{\log x} \quad \le \quad \pi(x) \quad \le \quad 1.11\;\frac{x}{\log x}, \tag{156} $$

on the prime counting function $\pi(x)$.

Considering hypergeometric series such that their coefficients are ratio of factorials, a paper by Rodrigues and Villegas [127] gives the conditions of these factorials for the hypergeometric series to be algebraic (all the coefficients are thus integers). A simple example is, for instance the algebraic function:

$$ _3F_2\Big([\frac{1}{4}, \frac{1}{2}, \frac{3}{4}], [\frac{1}{3}, \frac{2}{3}]; \; \frac{256}{27}\cdot x\Big) \quad = \quad \sum_{n=0}^{\infty} \binom{4\,n}{n} \cdot x^n. \tag{157} $$

Along this line it is worth recalling Delaygue's Thesis [129] (see also Bober [130]) which gives some results♯ for series expansions¶ such that their coefficients are *ratio of factorials*:

$$ _2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1]; \; 27\,x\Big) \quad = \quad \sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} \cdot x^n, \tag{158} $$

$$ _4F_3\Big([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}], [1, 1, 1]; \; 256\cdot x\Big) \quad = \quad \sum_{n=0}^{\infty} \frac{((2n)!)^4}{(n!)^8} \cdot x^n, \tag{159} $$

$$ _4F_3\Big([\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}], [1, 1, 1]; \; 2^8\, 3^3\cdot x\Big) \quad = \quad \sum_{n=0}^{\infty} \frac{(6n)!\;(2n)!}{(3n)!\,(n!)^5} \cdot x^n. \tag{160} $$

These ratio of factorials are integer numbers. The series expansion of (160) reads:

$$ 1 \; + 240\,x \; + 498960\,x^2 \; + 1633632000\,x^3 \; + 6558930378000\,x^4 $$
$$ + 29581300719210240\,x^5 + 143836335737833939200\,x^6 \quad + \quad \cdots $$

♯ Necessary and sufficient conditions for the integrality of the mirror maps series.
¶ These series are not algebraic functions.

8.9. More Hadamard products: Batyrev and van Straten examples [90]

8.9.1. A first auto-adjoint Calabi-Yau ODE

An order-four operator has been found by Batyrev and van Straten [90]

$$B_1 \;=\; \theta^4 \; - 3\,x \cdot (7\,\theta^2 \,+ 7\,\theta \,+ 2) \cdot (3\,\theta \,+ \,1) \cdot (3\,\theta \,+ \,2) \\ \qquad - 72\,x^2 \cdot (3\,\theta \,+ \,5) \cdot (3\,\theta \,+ \,4) \cdot (3\,\theta \,+ \,2) \cdot (3\,\theta \,+ \,1), \qquad (161)$$

which is conjugated to its adjoint: $B_1 \cdot x \,=\, x \cdot adjoint(B_1)$.

Operator (161) is a Calabi-Yau operator: it is MUM, and it is such that its exterior square is of *order five*. Its has a solution analytical at $x \,=\, 0$ which is actually the Hadamard product of the previous selected hypergeometric ${}_2F_1$:

$${}_2F_1\Bigl([\frac{1}{3},\,\frac{2}{3}],[1];\;27\,x\Bigr) \,\star\, \Bigl(\frac{1}{1\,+4\,x}\,\cdot\,{}_2F_1\Bigl([\frac{1}{3},\,\frac{2}{3}],\,[1];\,\frac{27 \cdot x}{(1\,+4\,x)^3}\Bigr)\Bigr). \qquad (162)$$

8.9.2. A second auto-adjoint Calabi-Yau ODE

A second example [90] (see† page 34) of order-four operator corresponds to Calabi-Yau 3-folds in $P_1 \times P_1 \times P_1 \times P_1$:

$$B_2 \;=\; \theta^4 \; - 4\,x \cdot (5\,\theta^2 \,+ 5\,\theta \,+ 2) \cdot (2\,\theta \,+ \,1)^2 \\ \qquad + 64\,x^2 \cdot (2\,\theta \,+ \,3) \cdot (2\,\theta \,+ \,1) \cdot (2\,\theta \,+ \,2)^2, \qquad (163)$$

corresponding to the series-solution with coefficients:

$$\binom{2n}{n} \cdot \sum_{k=0}^{n} \binom{n}{k}^2 \cdot \binom{2k}{k} \cdot \binom{2n-2k}{n-k} \qquad (164)$$

$$= \; \binom{2n}{n}^2 \cdot {}_2F_1\Bigl([\frac{1}{2},\,-n,\,-n,\,-n],\;[1,\,1,\,-\frac{2\,n\,-1}{2}];\,1\Bigr).$$

Its Wronskian reads:

$$W \;=\; \frac{1}{(1\,-64\,x)^2 \cdot\, (1\,-16\,x)^2 \cdot\, x^6}, \qquad x^3 \cdot W^{1/2} \;=\; \frac{1}{(1\,-64\,x)\,(1\,-16\,x)}. \qquad (165)$$

This operator is also a Calabi-Yau operator: it is MUM, and it is such that its exterior square is *order five*. This order five property is a consequence of B_2 being conjugated to its adjoint: $B_2 \cdot x \,=\, x \cdot adjoint(B_2)$.

The series-solution of (163) can be written as an Hadamard product

$$\mathcal{S} \;=\; (1\,-4\,x)^{-1/2} \,\star\, HeunG(4, 1/2, 1/2, 1/2, 1, 1/2;\,16\,x)^2 \qquad (166)$$
$$=\; 1 + 8\,x \,+ 168\,x^2 \,+ 5120\,x^3 \,+ 190120\,x^4 \,+ 7939008\,x^5 \,+ 357713664\,x^6 \;+\; \cdots,$$

the modular form character of $HeunG(4, 1/2, 1/2, 1/2, 1, 1/2;\,16\,x)$ being illustrated with identities (A.2) in Appendix A. Its nome reads:

$$q \;=\; x \;\, + 20\,x^2 \,+ 578\,x^3 \,+ 20504\,x^4 \,+ 826239\,x^5 \,+ 36224028\,x^6 \,+ 1684499774\,x^7 \\ + 81788693064\,x^8 \,+ 4104050140803\,x^9 \,+ 211343780948764\,x^{10} \;\; + \;\cdots \qquad (167)$$

Its *mirror map* reads:

$$x(q) \;=\; q \;\; - 20\,q^2 \,+ 222\,q^3 \,- 2704\,q^4 \,+ 21293\,q^5 \,- 307224\,q^6 \,+ 80402\,q^7 \\ - 67101504\,q^8 \,- 1187407098\,q^9 \,- 37993761412\,q^{10} \;\; + \;\cdots \qquad (168)$$

† There is a small misprint in [90] page 34: $(2\,\theta\,+\,1)$ must be replaced by $(2\,\theta\,+\,1)^2$ in the $4\,x$ term.

The Yukawa coupling of (163) reads:

$$K(q) \;=\; K^\star(q) \;=\; 1 \;+4\,q \;+164\,q^2 \;+5800\,q^3 \;+196772\,q^4 \;+6564004\,q^5 \;+222025448\,q^6 \;+7574684408\,q^7 \;+259866960036\,q^8 \;+\;\cdots \tag{169}$$

The equality of the Yukawa coupling with the "adjoint" Yukawa coupling, $K(q) = K^\star(q)$, is a straight consequence of relation $B_2 \cdot x \,=\, x \cdot adjoint(B_2)$.

Do note that recalling Batyrev and van Straten [90], (see step2 page 496), and following Morrison [89], one can also write the Yukawa coupling as:

$$K(q) \;=\; \frac{x(q)^3 \, \cdot W_4^{1/2}}{y_0^2} \cdot \Big(\frac{q}{x(q)} \cdot \frac{dx(q)}{dq}\Big)^3 \;=\; \frac{W_4^{1/2}}{y_0^2} \cdot \Big(q \cdot \frac{dx(q)}{dq}\Big)^3, \tag{170}$$

where W_4 is the Wronskian (165). From this alternative expression for the Yukawa coupling it is obvious that if the analytic series $y_0(x)$, *as well as* the nome (167) are series with integer coefficients, then, the mirror map (168) is also a series with integer coefficients, and, therefore, y_0 seen as a function of the nome q, as well as $x^3\, W_4^{1/2}$, and, consequently, the Yukawa coupling is a series with integer coefficients (as a series in q or in x).

The globally bounded character of the analytic series $y_0(x)$ together with the nome, thus yields the globally bounded character of the mirror map, Yukawa coupling, that we associate with the modularity. Similar results can be found in Delaygue's thesis [129], in a framework when the coefficients of hypergeometric series are ratio of factorials.

In contrast the globally bounded character of the analytic series $y_0(x)$, together with the globally bounded character of the Yukawa coupling (seen for instance as a series in x) does not imply that the nome, or the mirror map, are globally bounded as can be seen on example (113) (see (114) and (115)).

8.9.3. An operator non trivially homomorphic to B_2

Let us, now, introduce the order-four operator

$$\mathcal{B}_2 \;=\; 256\,x^2 \cdot \,\theta^2\,(2\,\theta+3)\,(2\,\theta+1) \;-\,4\,x \cdot \,(2\,\theta+1)\,(2\,\theta-1)\,(5\,\theta^2-5\,\theta+2) \;+\,(\theta-1)^4. \tag{171}$$

This operator is non-trivially‡ homomorphic to the Calabi-Yau operator (163):

$$\mathcal{B}_2 \cdot \, x \cdot (2\,\theta\,+1) \;=\; x \cdot (2\,\theta\,+1) \cdot \, B_2. \tag{172}$$

As a consequence of the previous intertwining relation, one immediately finds that the series-solution analytic at $x\,=\,0$ of this new MUM operator (171) is nothing but the action of the order-one operator $x \cdot (2\,\theta\,+1)$ on the series (166), and reads:

$$x \cdot (2\,\theta\,+1)[\mathcal{S}] \;=\; x \;+24\,x^2 \;+840\,x^3 \;+35840\,x^4 \;+1711080\,x^5 \;+87329088\,x^6 \;+4650277632\,x^7 \;+254905896960\,x^8 \;+\;\cdots \tag{173}$$

It is obviously also a series with *integer coefficients* (the action of $x \cdot (2\,\theta\,+1)$ on the series with integer coefficient is straightforwardly a series with integer coefficients). More generally, the globally bounded series remain globally bounded series by operator equivalence (non trivial homomorphisms between operators: generically the intertwiner operators are not simple functions).

‡ The intertwiners between $\mathcal{B}_2$ and B_2 are operators not simple functions.

The exterior square of the order-four operator (171) is an order-six operator which is, in fact the LCLM of an order-five operator $\mathcal{E}_5$ and an order-one operator:

$$Ext^2(\mathcal{B}_2) \;=\; \mathcal{E}_5 \oplus \Big(D_x - \frac{d\ln(\rho(x))}{dx}\Big), \quad \text{where:} \quad \rho(x) \;=\; \frac{x}{(1-16\,x)\,(1-64\,x)}.$$

Operator $\mathcal{B}_2$ is non-trivially homomorphic to its adjoint:

$$\mathcal{B}_2 \cdot x^3 \cdot (2\,\theta+3) \cdot (2\,\theta+5) \;=\; x^3 \cdot (2\,\theta+3) \cdot (2\,\theta+5) \cdot adjoint(\mathcal{B}_2). \qquad (174)$$

The Yukawa coupling of this order-four operator (171), non-trivially homomorphic to (163), reads:

$$K(q) \;=\; 1 \;-4\,q \;-140\,q^2 \;-4040\,q^3 \;-64436\,\frac{q^4}{3} \;+1889332\,\frac{q^5}{3}$$
$$+\,88331368\,\frac{q^6}{5} \;+1652707624\,\frac{q^7}{9} \;-69295027684\,\frac{q^8}{63} \;+\;\cdots \qquad (175)$$

The Yukawa coupling series (175) is *not globally bounded*.

The "adjoint Yukawa coupling" of this order-four operator (171) reads:

$$K^\star(q) \;=\; 1 \;+12\,q \;+564\,q^2 \;+20440\,q^3 \;+865732\,q^4 \;+37162444\,q^5$$
$$+\,8255346664\,\frac{q^6}{5} \;+1121762648248\,\frac{q^7}{15} \;+72336859374772\,\frac{q^8}{21} \;+\;\cdots \qquad (176)$$

Again, the adjoint Yukawa coupling series (176) is *not globally bounded*.

On this example one sees that the Yukawa coupling of two non-trivially homomorphic operators are *not necessarily equal*. The Yukawa couplings of two homomorphic operators are equal when the two operators are *conjugated by a function* (trivial homomorphism). The modularity property is *not preserved* by (non-trivial) operator equivalence: it can depend on a condition that the exterior square of the order-four operators are of order *five*. The Calabi-Yau property is not preserved by operator equivalence.

To sum-up: All these examples show that the *integrality* (globally bounded series) is *far from identifying with modularity*. All these examples have to be taken into account if one has in mind to build new conjectures combining these globally boundedness of various series with the concept of diagonal of rational functions: for instance, can we imagine that being a diagonal of rational functions automatically yields that the nome or the Yukawa coupling are globally bounded series in q or x, etc ?

9. Conclusion

Seeking for the linear differential operators for the $\chi^{(n)}$'s, we first discovered that they were Fuchsian operators [14, 16], and, in fact, "special" Fuchsian operators, namely Fuchsian operators with rational exponents for all their singularities, and with Wronskians that are N-th roots of rational functions. Then we discovered that they were G-operators (or equivalently globally nilpotent [28]), and more recently, we accumulated results [33] indicating that they are "special" G-operators. There is, in fact, *two quite different kinds of "special character"* of these G-operators. On one side, we have the fact that one of their solutions is not only G-series, but is a *globally bounded* series. This special character has been addressed in this very paper, and we have seen that, in fact, this "*integrality* property [31]" is a consequence of *quite general mathematical assumptions* often satisfied in physics (the integrand is not only algebraic but analytic in all the variables (50)). However, we have also seen another special property of these G-operators, namely the fact that they seem to be quite systematically *homomorphic to their adjoints* [33]. We will show, in a forthcoming publication, that this last property amounts, on the associated linear differential systems, to having *special differential Galois groups*, and that their exterior or symmetric square, have *rational solutions*. This last property is a property of a more "physical" nature than the previous one, related to an underlying *Hamiltonian structure* [121], or as this is the case, for instance in the Ising model, related to the underlying isomonodromic structure in the problem, which yields the occurrence of some underlying Hamiltonian structure [121]. In general the *integrality* of G-operators *does not* imply the operator to be homomorphic to its adjoint, and conversely being homomorphic to its adjoint *does not* imply‡ integrality (and even does not imply† the operator to be Fuchsian). Interestingly, the $\chi^{(n)}$'s, as well as many important problems of theoretical physics, correspond to G-operators that present these two complementary "special characters" (integrality and, up to homomorphisms, self-adjointness), and, quite often, this is seen in the framework of the emergence of "modularity".

Nomes, mirror maps, and Yukawa couplings are *not D-finite* functions: they are solutions of quite involved *non-linear* (higher order Schwarzian) ODEs (see for instance Appendix D in [28]). Therefore, the question of the series integrality of the nomes, mirror maps, Yukawa couplings, and other pullback-invariants (see Appendix J) requires to address the very difficult question of series-integrality for (involved) *non-linear* ODEs, or, equivalently, the problem of non-linear recursions with integer sequence solutions. Note, however, as seen in Section 8.9.2, in particular in (170), that the integrality of the series $y_0(x)$ and of the nome $q(x)$ are *sufficient to ensure, provided the operator is conjugated to its adjoint* (see (**??**)), the integrality of the other quantities such as the Yukawa coupling, mirror maps. However the integrality of the nome remains an involved problem. These questions will certainly remain open for some time.

In contrast, and more modestly, we have shown that a *very large sets of problems in mathematical physics* (see sections (4.4), (5) and (6.1)) *actually correspond to diagonals of rational functions*. In particular, we have been able to show that the $\chi^{(n)}$'s n-fold integrals of the susceptibility of the two-dimensional Ising model are actually *diagonals of rational functions for any value of the integer* n, thus proving that the $\chi^{(n)}$'s are *globally bounded for any value of the integer* n. As can be seen in the "ingredients" of our simple demonstration (see (4.4)), no elliptic curves, and their modular forms [132], no Calabi-Yau [118], or Frobenius manifolds [121], or Shimura curves, or arithmetic lattice assumption [119, 120] is required to prove the result. We just need to have a n-fold integral such that its integrand is not only algebraic, but *analytic in all the variables*.

Acknowledgments We would like to thank A. Enge and F. Morain for

‡ See Appendix N which gives an example of a (hypergeometric) family of order-four operators satisfying the Calabi-Yau condition that their exterior square is of order five, and, even, a family of self-adjoint order-four operator, the corresponding hypergeometric solution-series being *not globally bounded*. See also Appendix O.

† For instance the operator $D_x^n - x\, D_x - 1/2$ (see page 74 of [131]) with an irregular singularity is self-adjoint.

interesting and detailed discussions on Fricke and Atkin-Lehner involutions. S. B. would like to thank the LPTMC and the CNRS for kind support. A. B. was supported in part by the Microsoft Research–Inria Joint Centre. As far as physicists authors are concerned, this work has been performed without any support of the ANR, the ERC or the MAE.

Appendix A. Modular forms and series integrality

First example: The generating function of the integers

$$\sum_{k=0}^{n} \binom{n}{k}^2 \cdot \binom{2k}{k} \cdot \binom{2n-2k}{n-k} \tag{A.1}$$
$$= \quad \binom{2n}{n} \cdot {}_2F_1\Big([\frac{1}{2}, \, -n, \, -n, \, -n], \, [1, \, 1, \, -\frac{2\,n\,-1}{2}]; \, 1\Big),$$

is nothing else but the expansion of the square of a HeunG function

$$HeunG\Big(4, \, \frac{1}{2}, \, \frac{1}{2}, \, \frac{1}{2}, \, 1, \, \frac{1}{2}; \, 16 \cdot x\Big) \, = \, 1 \, + 2\,x \, + 12\,x^2 \, + 104\,x^3$$
$$+ 1078\,x^4 \, + 12348\,x^5 \, + 150528\,x^6 \, + 1914432\,x^7 \, + \, \cdots \tag{A.2}$$

solution of the order-two operator

$$H_{diam} \, = \, \theta^2 \, - 2 \cdot x \cdot (10\,\theta^2 \, + 5\,\theta \, + 1) \, + 16\,x^2 \cdot (2\,\theta \, + 1)^2. \tag{A.3}$$

which corresponds to the *diamond lattice* [48]. This HeunG function (A.2) is *actually a modular form*† which can be written in two different ways:

$$HeunG(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}; \, 16\,x)$$
$$= \, (1 \, - 4\,x)^{-1/2} \, \cdot \, {}_2F_1\Big([\frac{1}{6}, \, \frac{1}{3}], \, [1]; \, \frac{108\,x^2}{(1 \, - 4\,x)^3}\Big) \tag{A.4}$$
$$= \, (1 \, - 16\,x)^{-1/2} \, \cdot \, {}_2F_1\Big([\frac{1}{6}, \, \frac{1}{3}], \, [1]; \, -\frac{108\,x}{(1 \, - 16\,x)^3}\Big).$$

These two pullbacks are *related by an "Atkin" involution* $x \leftrightarrow 1/64/x$. The associated modular curve, relating these two pullbacks (A.4) yielding *the modular curve*:

$$4 \cdot y^3\,z^3 \, - 12\,y^2\,z^2 \cdot (y+z) \, + 3\,y\,z \cdot (4\,y^2 \, + \, 4\,z^2 \, - 127\,y\,z)$$
$$- 4 \cdot (y \, + z) \cdot (y^2 \, + z^2 \, + 83\,y\,z) \, + 432\,y\,z \, = \, 0, \tag{A.5}$$

which is $(y, \, z)$-symmetric and is *exactly the rational modular curve* in eq. (27) already found for the order-three operator F_3 in [33] for the five-particle contribution $\tilde{\chi}^{(5)}$ of the magnetic susceptibility of the Ising model.

This result in [46, 48] can be rephrased as follows. One introduces the order-three operator which has the following ${}_3F_2$ solution

$$\frac{1}{(4 \, - \, x^2)^3} \, \cdot \, {}_3F_2\Big([\frac{1}{3}, \, \frac{1}{2}, \, \frac{2}{3}], \, [1, \, 1], \, \frac{27\,x^4}{(4 \, - \, x^2)^3}\Big), \tag{A.6}$$

† Generically HeunG functions are far from being modular forms. They are even far from being solutions of globally nilpotent operators (they generically have no integral representations [133, 134]). There is a relation between these operators being finite-gap [135] and their globally nilpotence.

associated with the *Green function of the diamond lattice*. Along a *modular form line* lets us note that this hypergeometric function actually has *two* pullbacks:

$$ {}_3F_2\Big([\frac{1}{3}, \frac{1}{2}, \frac{2}{3}], [1, 1], \frac{27\,x^4}{(4\,-\,x^2)^3}\Big) \tag{A.7}$$
$$ = \quad \frac{x^2\,-4}{4\cdot(x^2-1)}\cdot\; {}_3F_2\Big([\frac{1}{3}, \frac{1}{2}, \frac{2}{3}], [1, 1], \frac{27\,x^2}{4\cdot\,(x^2-1)^3}\Big). $$

These two pullbacks related by the "Atkin" involution $x \rightarrow 2/x$:

$$ u(x)\,=\,\frac{27\,x^4}{(4\,-\,x^2)^3}, \qquad v(x)\;=\;u(\frac{2}{x})\;=\;\frac{27\,x^2}{4\cdot\,(x^2-1)^3}, \tag{A.8}$$

corresponding, again, to the modular curve (A.5).

Second example. The HeunG function

$$ HeunG(-3,0,1/2,1,1,1/2;\,12\cdot x) \tag{A.9}$$
$$ =\quad(1\,+4\,x)^{-1/4}\cdot\,HeunG\Big(4,\,\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2},\,1,\,\frac{1}{2},\,\frac{16\,x}{1\,+4\,x}\Big)$$
$$ =\quad 1\quad+6\,x^2\,+24\,x^3\,+252\,x^4\,+2016\,x^5\,+19320\,x^6\,+183456\,x^7 $$
$$ +\,1823094\,x^8\,+18406752\,x^9\,+189532980\,x^{10}\;+\;\cdots $$

is solution of

$$ Heun_{fcc}\;=\;\theta^2\;-2\,x\cdot\theta\cdot(4\,\theta\,+\,1)\;-24\cdot x^2\cdot(2\,\theta\,+\,1)\cdot(\theta\,+\,1), \tag{A.10}$$

The square of (A.9) is actually the solution of an order-three operator (see equation (19) in [48]) emerging for lattice Green functions of the face-centred cubic (fcc) lattice which is thus the symmetric square of (A.10). This hypergeometric function with a polynomial pull-back can also be written:

$$ HeunG(-3,\,0,\,1/2,\,1,\,1,\,1/2;\,12\cdot x)$$
$$ =\quad {}_2F_1\Big([\frac{1}{6},\,\frac{1}{3}],[1];\,108\cdot x^2\cdot(1+4\,x)\Big) \tag{A.11}$$
$$ =\quad(1-12\,x)^{-1/2}\;\cdot\,{}_2F_1\Big([\frac{1}{6},\,\frac{1}{3}],[1];\,-\,\frac{108\cdot x\cdot(1+4\,x)^2}{(1-12\,x)^3}\Big),$$

where the involution $x\,\leftrightarrow\,-1/4\cdot(1+4\,x)/(1-12\,x)$ takes place. The modular curve relating these two pullbacks reads *exactly the rational curve* (A.5) already obtained in [33].

Third example. The HeunG function $HeunG(1/9,1/12,1/4,3/4,1,1/2;\,4\,x)$ is solution of the order-two operator corresponding to the simple cubic lattice Green function

$$ H_{sc}\;=\;\theta^2\;\;-\,x\cdot(40\,\theta^2+20\,\theta\,+3)\;\;+9\cdot x^2\cdot(4\,\theta\,+3)\cdot(4\,\theta\,+1).$$

The square of this HeunG function is a series with integer coefficients which identifies with the Hadamard product of $(1\,-4\,x)^{-1/2}$ with a modular form :

$$ HeunG(1/9,1/12,1/4,3/4,1,1/2;\,4\,x)^2 \tag{A.12}$$
$$ =\quad(1-4\,x)^{-1/2}\,\star\,HeunG(1/9,1/3,1,1,1,1;\,x)$$
$$ =\quad 1\quad+6\,x\;+90\,x^2\;+1860\,x^3\;+44730\,x^4\;+1172556\,x^5\;+32496156\,x^6$$
$$ +\,936369720\,x^7\;+27770358330\,x^8\;+842090474940\,x^9\;\;+\;\cdots$$

The HeunG function $HeunG(1/9,1/12,1/4,3/4,1,1/2;\,4\,x)$ is globally bounded: the series of $HeunG(1/9,1/12,1/4,3/4,1,1/2;\,8\,x)$ is a series with integer coefficients.

One can also write this HeunG function in terms of a ${}_2F_1([1/6, 1/3], [1], x)$ hypergeometric function up to a simple algebraic pullback (with a square root), or in terms of a ${}_2F_1([1/8, 3/8], [1], x)$ hypergeometric function:

$$HeunG(1/9, 1/12, 1/4, 3/4, 1, 1/2; 4\,x) \,\, = \,\, C_2^{1/4} \cdot {}_2F_1\Bigl([1/8,\, 3/8],\, [1];\, P_2\Bigr), \quad \text{with:}$$

$$C_2 \,\, = \,\, \frac{1}{9 \cdot (1\, + 12\,x)^2} \cdot \Bigl(5\, - 36\,x\, + 4 \cdot (1 - 36\,x)^{1/2}\Bigr), \qquad P_2 \,\, = \,\, \frac{128 \cdot x}{(1\, + 12\,x)^4} \cdot p_2,$$

$$p_2 \,\, = \,\, (1\, - 42\,x\, + 352\,x^2\, - 288\,x^3) \,\, + (1\, - 4\,x) \cdot (1\, - 20\,x) \cdot \, (1\, - 36\,x)^{1/2}.$$

Do note that taking the Galois conjugate (changing $(1 - 36\,x)^{1/2}$ into $-(1 - 36\,x)^{1/2}$) gives the series expansion of $3^{-1/2} \cdot HeunG(1/9, 1/12, 1/4, 3/4, 1, 1/2; 4\,x)$. This shows that there exists an identity for ${}_2F_1([1/8, 3/8], [1], x)$ with *two different pullbacks*, namely the previous P_2 and its Galois conjugate, these two pullbacks being related by a (symmetric genus zero) modular curve:

$$\begin{aligned} &5308416 \cdot y^4\, z^4 \,\, + 442368 \cdot y^3\, z^3 \cdot (y + z) \,\, + 512\, y^2\, z^2 \cdot (27\, y^2\, + 27\, z^2\, - 27374\, x\, y) \\ &\quad + 192\, y\, z \cdot (y\, + z) \cdot (y^2\, + z^2\, + 10718\, y\, z) \,\,\, + y^4\, + z^4\, + 3622662\, y^2\, z^2 \\ &\quad\quad - 19332 \cdot y\, z \cdot \, (y^2\, + \, z^2) \,\, + 79872 \cdot y\, z \cdot (y\, + z) - 65536 \cdot y\, z \,\,\, = \,\,\, 0. \qquad (A.13) \end{aligned}$$

Revisiting the examples. In a recent paper [136] corresponding to spanning tree generating functions and Mahler measures, a result from Rogers (equation (36) in [136]) is given where the two following ${}_5F_4$ hypergeometric functions take place:

$${}_5F_4\Bigl([\frac{5}{4},\, \frac{3}{2},\, \frac{7}{4},\, 1,\, 1],\, [2,\, 2,\, 2,\, 2],\, \frac{256\,x^3}{9 \cdot\, (x + 3)^4}\Bigr),$$
$${}_5F_4\Bigl([\frac{5}{4},\, \frac{3}{2},\, \frac{7}{4},\, 1,\, 1],\, [2,\, 2,\, 2,\, 2],\, \frac{256\,x}{9 \cdot\, (1\, +\, 3\,x)^4}\Bigr). \qquad (A.14)$$

The corresponding order-five linear differential operators (annihilating these two ${}_5F_4$ hypergeometric functions) are actually homomorphic (the intertwiners being order-four operators). The relation between these two pullbacks $y \, = \, 256\,x^3/9/(x\, +\, 3)^4$ and $z \, = \, 256\,x/9/(1\, +\, 3\,x)^4$, remarkably *gives, again, the previous* $(y,\, z)$*-symmetric modular curve* (A.13).

The order-five linear differential operator, corresponding to the first ${}_5F_4$ hypergeometric function, factorizes in an order-one operator, an order-three operator and an order-one operator, the order-three operator being, in fact, exactly the symmetric square of an order-two operator:

$$L_1 \,\, \cdot \,\, Sym^2(W_2) \,\, \cdot \,\, \frac{x^4}{(x\, - 9)\, (x\, + 3)^4} \,\, \cdot \,\, R_1,$$

where the order-one operators read respectively

$$L_1 = \, D_x \, - \, \frac{d}{dx} \ln\Bigl(\frac{x - 9}{(9\,x^2\, + 14\,x\, + 9) \cdot \, (x + 3)^4}\Bigr), \qquad R_1 \,\, = \,\, D_x \,\, - \, \frac{d}{dx} \ln\Bigl(\frac{(x + 3)^4}{x^3}\Bigr),$$

and where the order-two operator W_2 reads:

$$W_2 \,\, = \,\,\, D_x^2 \,\,\, + 3\, \frac{(6 \cdot \, x^2\, + 7\,x\, + 3)}{(9\,x^2\, + 14\,x\, + 9) \cdot \, x} \cdot D_x \,\,\, + \,\, \frac{3}{4} \cdot \, \frac{3\,x\, + 2}{(9\,x^2\, + 14\,x\, + 9) \cdot \, x}. \qquad (A.15)$$

We have a similar result for the order-five linear differential operator corresponding to the second ${}_5F_4$ hypergeometric function.

Another solution of this order-five linear differential operator reads:

$$\frac{(x+3)^4}{x^3} \cdot \int \frac{x-9}{(x+3)\cdot x} \cdot {}_3F_2\Big([\frac{1}{4}, \frac{1}{2}, \frac{3}{4}], [1, 1], \frac{256\, x^3}{9\cdot (x+3)^4}\Big) \cdot dx. \qquad (A.16)$$

The expansion of the ${}_3F_2$ hypergeometric function in (A.16) is globally bounded (change $x \rightarrow 9\,x$ to get a series with integer coefficients).

Recalling the two previous pullbacks we have, in fact, the following identity:

$$3 \cdot (1+3\,x) \cdot {}_3F_2\Big([\frac{1}{4}, \frac{1}{2}, \frac{3}{4}], [1, 1], \frac{256\, x^3}{9\cdot (x+3)^4}\Big)$$
$$= (x+3) \cdot {}_3F_2\Big([\frac{1}{4}, \frac{1}{2}, \frac{3}{4}], [1, 1], \frac{256\, x}{9\cdot (1+3\,x)^4}\Big). \qquad (A.17)$$

However this ${}_3F_2$ hypergeometric function is nothing but the square of a ${}_2F_1$ hypergeometric function

$${}_3F_2\Big([\frac{1}{4}, \frac{1}{2}, \frac{3}{4}], [1, 1], x\Big) = {}_2F_1\Big([\frac{1}{8}, \frac{3}{8}], [1], x\Big)^2. \qquad (A.18)$$

Thus, the previous identity (A.17) is nothing but the identity on a ${}_2F_1$ hypergeometric function with *two different pullbacks*:

$$(1+3\,x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{8}, \frac{3}{8}], [1], \frac{256\, x^3}{9\cdot (x+3)^4}\Big)$$
$$= \Big(1 + \frac{x}{3}\Big)^{1/2} \cdot {}_2F_1\Big([\frac{1}{8}, \frac{3}{8}], [1], \frac{256\, x}{9\cdot (1+3\,x)^4}\Big). \qquad (A.19)$$

The expansion of (A.19) is globally bounded. One gets a series with *positive integer* coefficients using the simple rescaling $x \rightarrow 36\cdot x$. Note that the two pullbacks can be exchanged by the simple "Atkin" involution $x \leftrightarrow 1/x$, being related by the modular curve occurring for the simple cubic lattice, namely (A.13).

We have a similar result for the other ${}_5F_4$ hypergeometric functions popping out in [136].

For instance, for the diamond lattice one gets an expression (see eq. (50) in [136]) where the two following ${}_5F_4$ hypergeometric functions take place‡:

$${}_5F_4\Big([\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, 1, 1], [2, 2, 2, 2], \frac{-27\, x^2}{4\cdot (1-x^2)^3}\Big),$$
$${}_5F_4\Big([\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, 1, 1], [2, 2, 2, 2], \frac{27\, x^4}{(4-x^2)^3}\Big). \qquad (A.20)$$

These two pullbacks can be exchanged by the simple "Atkin" involution $x \leftrightarrow 2/x$. These two pullbacks have been seen to be related by the (genus-zero) $(y,\, z)$-symmetric modular curve (A.5):

$$4\,y^3\,z^3 - 12\,y^2\,z^2\cdot (y+z) + 3\,y\,z\,\big(4\,y^2 + 4\,z^2 - 127\,y\,z\big)$$
$$- 4\cdot (y+z)\cdot (y^2 + z^2 + 83\,y\,z) + 432\,y\,z = 0. \qquad (A.21)$$

Similarly to (A.17) we have an identity between two ${}_3F_2$ hypergeometric functions (namely ${}_3F_2([2/3,\, 1/2,\, 1/3], [1,\, 1], z))$ with the two pullbacks (A.20), and these ${}_3F_2$

‡ Note a small misprint in eq. (50) of [136]: one should read $-27z^2/4/(1-z^2)^3$ instead of $-27z^4/4/(1-z^2)^3$.

hypergeometric functions being the square of ${}_2F_1$ hypergeometric functions, one finds that the "deus ex machina" is the identity similar to (A.19):

$$(1 - x^2)^{1/2} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{1}{6}], [1], \frac{27\,x^4}{(4 - x^2)^3}\Big)$$
$$= (1 - \frac{x^2}{4})^{1/2} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{1}{6}], [1], \frac{-27\,x^2}{4 \cdot (1 - x^2)^3}\Big). \qquad (A.22)$$

The series expansion of (A.22) is globally bounded. Rescaling the x variable as $x \to 4\,x$, the series expansion becomes a series with *positive integer* coefficients (up to the first constant term).

For the face-centred cubic lattice one gets an expression (see eq. (52) in [136]) where the two following ${}_5F_4$ hypergeometric functions take place†:

$${}_5F_4\Big([\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, 1, 1], [2, 2, 2, 2], \frac{x \cdot (x + 3)^2}{(x - 1)^3}\Big),$$
$${}_5F_4\Big([\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, 1, 1], [2, 2, 2, 2], \frac{x^2 \cdot (x + 3)}{4}\Big). \qquad (A.23)$$

This example is nothing but the previous diamond lattice example (A.20) with the change of variable $x \to -3\,x^2/(x^2 - 4)$ in (A.23). Therefore, the two pullbacks in (A.23) are, again, related by the modular curve (A.5). The two pullbacks in (A.23) can actually be seen directly in the following identity (equivalent to (A.22)):

$${}_2F_1\Big([\frac{1}{3}, \frac{1}{6}], [1], \frac{x \cdot (x + 3)^2}{(x - 1)^3}\Big) = (1 - x^2)^{1/2} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{1}{6}], [1], \frac{x^2 \cdot (x + 3)}{4}\Big).$$

Finally, the equation (17) of [136] on Mahler measures, the two following ${}_4F_3$ hypergeometric functions take place:

$${}_4F_3\Big([\frac{5}{3}, \frac{4}{3}, 1, 1], [2, 2, 2], \frac{27\,x}{(x - 2)^3}\Big),$$
$${}_4F_3\Big([\frac{5}{3}, \frac{4}{3}, 1, 1], [2, 2, 2], \frac{27\,x^2}{(x + 4)^3}\Big). \qquad (A.24)$$

These two previous pullbacks can be exchanged by an "Atkin" involution $x \leftrightarrow -8/x$ and are related by the (genus-zero) $(y,\, z)$-symmetric modular curve:

$$8\,y^3\,z^3 - 12\,y^2 z^2 \cdot (y + z) + 3\,y\,z \cdot (2\,y^2 + 2\,z^2 + 13\,y\,z)$$
$$- (y + z) \cdot (y^2 + z^2 + 29\,y\,z) + 27\,y\,z = 0. \qquad (A.25)$$

The underlying identity on ${}_2F_1$ hypergeometric functions with the two pullbacks (A.24) read:

$$- 2 \cdot (x - 2) \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \frac{27\,x^2}{(x + 4)^3}\Big)$$
$$= (x + 4) \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \frac{27\,x}{(x - 2)^3}\Big). \qquad (A.26)$$

The series expansion of (A.26) is globally bounded. Rescaling the x variable as $x \to -8\,x$, the series expansion becomes a series with *positive integer* coefficients.

† There is one more misprint in [136]: the pullback $-x\,(x + 3)/(x - 1)^3$ must be changed into $x\,(x + 3)/(x - 1)^3$.

Appendix B. Another logarithmically bounded series

Let us display other logarithmically bounded series than (2). The hypergeometric function ${}_2F_1([N/3, 1/6], [7/6], 9\,x)$ is *not globally bounded* but is such that the order-one operator $6\,\theta\,+1$ acting on it, is a series with integer coefficients for every integer value of N:

$$(6\,\theta\,+1)\,[{}_2F_1\Big([\frac{N}{3},\,\frac{1}{6}],\,[\frac{7}{6}],\,9\,x\Big)] \quad = \quad {}_1F_0\Big([\frac{N}{3}],\,[],\,9\,x\Big)$$

$$= \quad 1+3\,x+18\,x^2+126\,x^3+945\,x^4+7371\,x^5\,+58968\,x^6\,+\,\cdots \qquad \text{for } N\,=\,1,$$

$$= \quad 1+6\,x+45\,x^2+360\,x^3+2970\,x^4+24948\,x^5+212058\,x^6\,+\,\cdots \quad \text{for } N\,=\,2,$$

$$= \quad 1+9\,x+81\,x^2+729\,x^3+6561\,x^4+59049\,x^5+531441\,x^6\,+\,\cdots \quad \text{for } N\,=\,3,\;\cdots$$

Similarly

$$U \quad = \quad {}_3F_2\Big([\frac{1}{4},\,\frac{7}{12},\,\frac{1}{7}],\,[\frac{4}{3},\,\frac{8}{7}],\,64\,x\Big), \qquad \text{(B.1)}$$

which is such that the action of the order-one operator $7\,\theta\,+\,1$ changes it into a *globally bounded* function

$$(7\,\theta\,+\,1)(U) \quad = \quad {}_2F_1\Big([\frac{1}{4},\,\frac{7}{12}],\,[\frac{4}{3}],\,64\,x\Big) \quad = \quad 1\;\;+7\,x\;\,+190\,x^2\;\,+7068\,x^3$$
$$+\,303924\,x^4\;+14208447\,x^5\;+701448594\,x^6\;+35983401900\,x^7\;\;+\,\cdots$$

Appendix C. $\Phi_D^{(n)}(w)$ as diagonals

The family of simple integrals $\Phi_D^{(n)}(w)$ was introduced in §4 of [82], as a way to simplify the study of the singularities of the Ising integrals $\chi^{(n)}$. By definition, they are equal to

$$\Phi_D^{(n)}(w) \quad = \quad -\frac{1}{n!} \quad +\frac{2}{n!}\int_0^{2\pi}\frac{d\phi}{2\pi}\;\frac{1}{1\;\,-x^{n-1}(w,\phi)\cdot x(w,(n-1)\phi)}, \qquad \text{(C.1)}$$

where

$$x(w,\phi) \quad = \quad \frac{2w}{1\;-2w\,\cos(\phi)\;+\sqrt{(1\,-2w\cos(\phi))^2\,-4w^2}}. \qquad \text{(C.2)}$$

By an easy change of variables, it follows that

$$\Phi_D^{(n)}(w) \quad = \quad -\frac{1}{n!} \quad + \frac{2}{n!}\cdot\Psi_D^{(n)}(w), \qquad \text{(C.3)}$$

where

$$\Psi_D^{(n)}(w) \quad = \quad \frac{1}{\pi}\int_{-1}^{1} F_n(w,t)\cdot\frac{dt}{\sqrt{1-t^2}}, \qquad \text{(C.4)}$$

the algebraic function $F_n(w,\,t)$ being defined by

$$F_n(w,\,t) \quad = \quad \frac{1}{1\;\,-h(w,t)^{n-1}\cdot h(w,T_{n-1}(t))}, \qquad \text{(C.5)}$$

where

$$h(w,t) \quad = \quad \frac{2w}{1\;-2w\,t\;+\sqrt{(1-2wt)^2\;-4w^2}}, \qquad \text{(C.6)}$$

and where $T_m(t)$ is the m-th Chebyshev polynomial of the first kind, that is, the unique polynomial of degree m such that $\cos(mt) \,=\, T_m(\cos t)$.

In order to express $\Phi_D^{(n)}(w)$ as the diagonal of an algebraic function in two variables, it is sufficient to use the following general result:

If $F(w,t)$ is a bivariate power series, then the univariate power series

$$\Psi(w) \quad = \quad \frac{1}{\pi} \cdot \int_{-1}^{1} \frac{F(w,\, t)}{\sqrt{1-t^2}} \cdot dt \tag{C.7}$$

is the diagonal of the generalised power series§

$$G(w,t) \quad = \quad \frac{F(w\,t,\, 1/t)}{\sqrt{1-t^2}}. \tag{C.8}$$

The only non-trivial point in the proof of this fact is the classical integral evaluation

$$\frac{1}{\pi} \cdot \int_{-1}^{1} \frac{dt}{(1-ut)\cdot\sqrt{1-t^2}} \quad = \quad \frac{1}{\sqrt{1-u^2}}, \qquad \text{for} \quad |u| < 1. \tag{C.9}$$

Expanded proof: Letting $F(w,t) \,=\, \sum_{\ell \ge 0} f_\ell(t)\cdot w^\ell$, the series $\Psi(w)$ is equal to

$$\Psi(w) \quad = \quad \sum_{\ell \ge 0} \frac{1}{\pi} \cdot \int_{-1}^{1} \frac{f_\ell(t)}{\sqrt{1-t^2}} \cdot w^\ell \; dt, \tag{C.10}$$

while the series $G(w,\, t)$ is equal to

$$G(w,\, t) \quad = \quad \sum_{\ell \ge 0} \frac{f_\ell(1/t)\; t^\ell}{\sqrt{1-t^2}} \cdot w^\ell. \tag{C.11}$$

It follows that the diagonal of G is equal to

$$\mathsf{Diag}(G)(w) \quad = \quad \sum_{\ell \ge 0} \left[t^0\right] \frac{f_\ell(1/t)}{\sqrt{1-t^2}} \cdot w^\ell. \tag{C.12}$$

To prove that $\Psi \,=\, \mathsf{Diag}(G)$, it thus suffices to show that for any power series f,

$$\frac{1}{\pi} \cdot \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} \cdot dt \quad = \quad \left[t^0\right] \frac{f(1/t)}{\sqrt{1-t^2}}. \tag{C.13}$$

By linearity, it thus suffices to prove that for any non-negative integer s,

$$\frac{1}{\pi} \cdot \int_{-1}^{1} \frac{t^s}{\sqrt{1-t^2}} \cdot dt \quad = \quad [t^s]\, \frac{1}{\sqrt{1-t^2}}. \tag{C.14}$$

This follows from the classical integral evaluation

$$\frac{1}{\pi} \cdot \int_{-1}^{1} \frac{dt}{(1-ut)\cdot\sqrt{1-t^2}} \quad = \quad \frac{1}{\sqrt{1-u^2}}, \qquad \text{for} \quad |u| < 1. \tag{C.15}$$

For example, when $n = \,2$, the previous construction shows that

$$\begin{aligned} \Phi_D^{(2)}(w) \quad &= \quad \frac{1}{4} \;\; + \frac{1}{4} \cdot {}_2F_1\Big([\frac{1}{2},\, \frac{1}{2}]\,[1],\, 16\,w^2\Big) \\ &= \quad \frac{1}{2} \;\; + w^2 \;\; + 9w^4 \;\; + 100w^6 \;\; + 1225w^8 \;\; + 15876w^{10} \;\;\; + \;\; \cdots \end{aligned} \tag{C.16}$$

is equal to the diagonal of the algebraic function

$$\frac{1 \;-2w \,+ \sqrt{(1-2w)^2 \;-4w^2t^2}}{2\,\cdot\,\sqrt{1-t^2}\cdot\sqrt{(1-2w)^2 \;-4w^2t^2}} \quad - \frac{1}{2}. \tag{C.17}$$

§ In the sense of [58].

Appendix D. Creative telescoping: computing ODEs for diagonals

The notion of diagonal of rational function is ubiquitous in combinatorics [137]. Its importance comes from the fact that many operations on power series with a combinatorial relevance (Hadamard products, constant terms, or positive parts, of Laurent power series, etc) *can be encoded as diagonals*. A classical result by Lipshitz [57] predicts that *diagonals of rational functions are D-finite*. The question is then: how to obtain algorithmically a differential equation satisfied by the diagonal $\mathsf{Diag}(f)$ of a given rational function $f(x_1, \ldots, x_n)$? The question can be reformulated in terms of computing a multiple integral with parameters, over an algebraic surface ("vanishing cycle" or "évanescent cycle" in Deligne's terminology [64]), and thus can be attacked from a geometric viewpoint.

A first answer to this algorithmic question is provided by Lipshitz's result [57]: if F denotes the rational function $F = f(x_1, x_2/x_1, \ldots, x_n/x_{n-1})/(x_1 \cdots x_{n-1})$, and if the following equality, called the *creative telescoping equation*,

$$L\left(x_n, \frac{\partial}{\partial x_n}\right)(F) \quad = \quad \frac{\partial R_1}{\partial x_1} \; + \; \cdots \; + \; \frac{\partial R_{n-1}}{\partial x_{n-1}}, \qquad \text{(D.1)}$$

admits a solution $(L, g_1, \ldots, g_{n-1})$, where P (called *telescoper*) is a linear differential operator with coefficients in $\mathbb{Q}[x_n]$, and where $R_1, \ldots, R_{n-1}$ are rational functions in $\mathbb{Q}(x_1, \ldots, x_n)$ (called *certificates*), then P annihilates the diagonal $\mathsf{Diag}(f)$ of f.

Several algorithms exist for solving equation (D.1). A common weakness of currently known algorithms for solving (D.1) is that they are not able to compute the telescoper L without computing the certificates $(R_1, \ldots, R_{n-1})$. This is unfortunate, since in practice only the telescoper is really needed, while the size of the certificates is much more important than that of the telescoper.

Lipshitz's initial argument requires the construction of a non-zero operator annihilating F which involves *all* the partial derivatives $\partial/\partial x_i$. This reduces the resolution of (D.1) to that of a linear system over $\mathbb{Q}$. The big practical issue with this approach is the size of the linear system, which is about several millions even for the simple rational function $f = 1/(1 - x_1 - x_2 - x_3)$. A much more efficient algorithm for solving equation (D.1) is Chyzak's extension [65, 138] of the Zeilberger's celebrated *creative telescoping method* [139], although the computational complexity of Chyzak's algorithm is not yet well understood. The most efficient implementation of Chyzak's algorithm is due to Koutschan [140].

More generally, for n-fold parameterised integrals of *D-finite functions*, Chyzak's creative telescoping algorithm delivers a system of PDEs. For two variables (anisotropic Ising model), one will get a system of PDEs corresponding to two "telescopers", that can be written in the following form:

$$P_1\Big(x,\, y,\, \frac{\partial}{\partial x}\Big) \;=\; \sum_{n=0}^{N} p_n(x,\, y) \cdot D_x^n, \qquad P_2\Big(x,\, y,\, \frac{\partial}{\partial x}\Big) \;=\; \sum_{n=0}^{M} q_n(x,\, y) \cdot D_y^n.$$

where the p_n's, and the q_n's, are polynomials of the two variables x and y.

Note that, in practice, the down-to-earth physicist's *guessing techniques* we have used in our various papers [14, 16, 26, 82], which amount to getting‡ the linear ODE

‡ Strictly speaking, the correctness of the linear ODEs obtained by "guessing" is not mathematically guaranteed. However, one may be convinced on the correctness of the ODE, since, in practice, one has longer series than what is used in the guessing. Also, some properties (as global nilpotence, expected known structures) are retrieved.

from the series expansion of the diagonal (or, in general, of the parameterised integral) is much more efficient† than the creative telescoping approach. This is moral, since time consuming computations are the price to pay in order to guarantee the correctness of the ODE.

Appendix E. Christol's theorem in more heuristic terms

Let us give a sketch in heuristic terms of how the main theorem in [49] is proved.

The first step is purely algebraic-geometric. The algebraic function F involved in the integral representation

$$f(x) \quad = \quad \int_C F(x; x_1, \ldots, x_n) \cdot \; dx_1 \, \cdots \, dx_n, \tag{E.1}$$

lives on a complex $(n+1)$-manifold V, or, more precisely, on a family of smooth complex n-manifolds V_x. One applies to it the so-called *embedded resolution of singularities* [142]. This process uses a succession of *blowing up* which is theoretically explicit but seems to be *inaccessible for computation*.

Roughly speaking, we so obtain a new family of manifold $\widetilde{V}$ with $\widetilde{V}_x = V_x$ for $x \neq 0$, and $\widetilde{V}_0$, a *divisor with normal crossing*, namely, a union of "smooth algebraic" n-manifolds D_i that meets "transversally". In particular, if non void, an intersection $D_{i_1} \cap \cdots \cap D_{i_m}$ of m distinct D_i's is of (complex) dimension $n - m + 1$. It is obvious that, at most, $n+1$ divisors D_i's can intersect at a given point of $\widetilde{V}_0$.

Moreover, if there are really $n+1$ divisors D_i intersecting at the point P of $\widetilde{V}_0$, then, choosing P as origin and equations $X_i = 0$ of D_i $(0 \le i \le n)$ as new variables, the equation of $\widetilde{V}$ becomes, at least locally, $X_0 ... X_n = x$. Applying the (algebraic) change of variables $(X_1, \ldots, X_n) = \varphi(x_1, \, \ldots, \, x_n)$ to (E.1) (x coming in the picture as a parameter), one gets

$$f(x) \quad = \quad \int_{\varphi(C)} F(x\, ; \, X_1, \, \ldots, \, X_n) \cdot \frac{dX_1}{X_1} \, \cdots \, \frac{dX_n}{X_n}, \tag{E.2}$$

for an algebraic function F. If we are lucky, the cycle $\varphi(C)$ is (homotopic to) C_P, the vanishing cycle around P, and (E.2) is an avatar of (53). Then

$$f \; = \; \mathsf{Diag}(\widetilde{F}) \qquad \text{with} \qquad \widetilde{F}(X_0, \, \ldots, \, X_n) \; = \; F(X_0 \cdots X_n \, ; \; X_1, \, \ldots, \, X_n), \tag{E.3}$$

(up to a multiplicative constant) and f is the diagonal of an algebraic function (in $n+1$ variables), hence, the diagonal of a rational function (in $2n+2$ variables).

Actually the computation of $\varphi(C)$ is inaccessible. To find hypothesis under which it is possible to conclude, we turn to the Picard-Fuchs equation L_V because it does not depend on the cycle C. Moreover it is stable under birational maps like the blowing up used for the *desingularisation*. All reasoning we will do from now do concern all solutions of L_V and cannot be done by considering only, for instance, the minimal order linear differential equation of f.

Reverting the process, we conclude from formula (E.2) that integration on C_P gives a solution of the Picard-Fuchs equation in the ring of diagonals of rational functions (proposition 11 in [49]). Let us recall that the Picard-Fuchs linear differential

† Serious programming improvements of the creative telescoping method have been developed recently [140, 141], and it is now possible to get the linear ODE for the isotropic Ising model $\tilde{\chi}^{(3)}$, from creative telescoping calculations.

equation is given by the derivation $\frac{\partial}{\partial x}$ acting (through derivation under the integral sign) on the space $H^n(\widetilde{V}_x)$ of n differentials modulo exact ones and a solution of this "differential module" is a $\mathbb{C}(x)$-linear application from this space to some function space (here the diagonals of rational function) that do commute with $\frac{\partial}{\partial x}$.

It is difficult to decide, a priori, whether, or not, such a solution is zero on the particular differential $F(x; X_1 \ldots X_n) \frac{dX_1}{X_1} \cdots \frac{dX_n}{X_n}$ we begin with. But we can assert that it is non zero for differentials $G(x\,; X_1, \ldots, X_n) \frac{dX_1}{X_1} \cdots \frac{dX_n}{X_n}$ such that $\widetilde{G}(0, \ldots, 0) \neq 0$. So, we consider the P-residue, for $P = D_1 \cap \cdots \cap D_n$, which, roughly speaking, associates to a given differential the coefficient of $\frac{dX_1}{X_1} \cdots \frac{dX_n}{X_n}$ it contains. Then the *Poincaré residue* map associates to a differential the family of its residues in all the point P of $\widetilde{V}_0$ which are the intersection of $n+1$ divisors D_i (this set could be void)§.

The last step is to connect the Poincaré residue and the monodromy filtration on the space of solutions of this differential module. This is more or less contained in [143]. Actually this paper shows how to compute subspaces of differential with logarithmic poles of given order by means of spaces built from the monodromy filtration. A by-product of this construction (cf theorem 12 in [49]) says that a differential of $H^n(\widetilde{V}_x)$ the Poincaré residue of which is 0 (i.e. it has a zero residue for each P) is in the kernel of any solution of maximal (monodromy) weight for L_V.

As a consequence, the kernel of a solution of maximal weight for L_V contains the intersection of the kernel of the solution corresponding to integrate on the vanishing cycles C_P. But the monodromy filtration is characterised by its "dual" filtration on $H^n(\widetilde{V}_x)$ given by corresponding kernels. So we conclude that the solution obtained by integrating on C is in the span of solutions obtained by integrating on the vanishing cycles C_P. Hence it takes its value *in the set of diagonals of rational functions* (in $2n+2$ variables).

When L_V is MUM a simpler argument is the following : by hypothesis the solution associated to C is of maximal weight and the differential, we started with, is not in its kernel because $f \neq 0$. So there is, at least, one point P such that integration on C_P of that differential is not zero and gives a diagonal of rational function g. In particular g is analytic (near zero) and one can conclude by unicity, up to a constant, of the analytic solution for L_V.

Appendix F. Other hypergeometric "blind spots" for Christol's conjecture

Let us give a list of ${}_3F_2$ that are not *algebraic hypergeometric* functions‡, that are not obviously Hadamard product of algebraic functions, but actually correspond to series with *integer* coefficients:

$$ {}_3F_2\Big([\frac{N_1}{9},\, \frac{N_2}{9},\, \frac{N_3}{9},],\, [\frac{M_1}{3},\, 1],\; 3^6\, x\Big), \tag{F.1} $$

§ It is rather easy to convince himself that differential $\frac{dX_1}{X_1} \cdots \frac{dX_i}{X_i^h} \cdots \frac{dX_n}{X_n}$ are exact ones for $h > 1$ and then are 0 in $H^n(\widetilde{V}_x)$. The naming logarithmic pole comes from this remark. It is much less obvious to prove that the Poincaré residue is actually well-defined on $H^n(V_x)$ but it does.

‡ The ${}_2F_1$ case is well-known.

where the four integers (N_1, N_2, N_3; M_1) read respectively:

[1, 2, 7; 2], [1, 2, 8; 2], [1, 4, 5; 1], [1, 4, 7; 1], [1, 4, 7; 2], [1, 4, 8; 2], [1, 5, 8; 1], [1, 7, 8; 1], [2, 4, 5; 1], [2, 4, 7; 1], [2, 5, 7; 2], [2, 5, 8; 1], [2, 5, 8; 2], [2, 7, 8; 1], [4, 5, 7; 2], [4, 5, 8; 2].

The series expansion of the first candidate reads:

$$ {}_3F_2\Bigl([\frac{1}{9}, \frac{2}{9}, \frac{7}{9},], [\frac{2}{3}, 1], 3^6 x\Bigr) \quad = \quad 1 \ + 21\,x \ + 5544\,x^2 \ + 2194500\,x^3 \ + 1032711750\,x^4 \ + 535163031270\,x^5 \ + 294927297193620\,x^6 \ + 169625328357359160\,x^7 + 100668944872954458000\,x^8 \ + \cdots $$

Other examples read for instance:

$$ {}_3F_2\Bigl([\frac{1}{7}, \frac{2}{7}, \frac{4}{7},], [\frac{1}{2}, 1], 7^4 x\Bigr), \qquad {}_3F_2\Bigl([\frac{1}{11}, \frac{2}{11}, \frac{6}{11},], [\frac{1}{2}, 1], 11^4 x\Bigr). \tag{F.2} $$

Do note that, even if these various hypergeometric functions look very much alike, the linear differential operators that annihilate them are *not equivalent* (no homomorphisms between these operators† or their symmetric powers).

One can also try to find, systematically, ${}_4F_3$ hypergeometric functions that are not *algebraic hypergeometric*, that are not obviously Hadamard product of algebraic functions, but actually correspond to series with integer coefficients. Note that some of these ${}_4F_3$ are deduced from the previous ${}_3F_2$, as a consequence of a Hadamard product by an algebraic function:

$$ {}_4F_3\Bigl([\frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{1}{4}], [\frac{2}{3}, 1, 1], 3^6\,2^3\,x\Bigr) \tag{F.3} $$

$$ = \quad {}_3F_2\Bigl([\frac{1}{9}, \frac{2}{9}, \frac{7}{9},], [\frac{2}{3}, 1], 3^6 x\Bigr) \star (1 - 2^3 x)^{-1/4} \quad = \quad 1 \ + 42\,x \ + 55440\,x^2 \ + 131670000\,x^3 \ + 402757582500\,x^4 \ + 1419252358928040\,x^5 \ + 5475030345102361680\,x^6 \ + 22492318540185824616000\,x^7 \ + \cdots $$

Appendix G. Proof of integrality of series (77)

Let us sketch the proof of the integrality of series (77), namely, the integrality of coefficients (79). For each power of the integer number $q = p^n$ a term like $4 + 9\,n$ is periodically divisible (period p) by q. In order to have the ratio (80) be an integer, one needs the numerator to be divisible by this factor q *before* the denominator. The case $p = 3$ is an easy one. The other prime p do not divide 9. One needs to find the first case of divisibility, namely the first integer n such that $4 + 9\,n = k\,q$ (this corresponds to the smallest k). If $d\,q = 1$, *mod.* 9 then $k = 4\,d$, *mod.* 9. In other words, the smallest k is the rest of $4\,d$, *mod.* 9. Consequently, we have replaced the calculations, for every integer q, by a *finite set of* calculations for $d = 1, 2, 4, 5, 7, 8$. Let us use this approach for the ratio (80).

Remark: The terms $n + 1$ are always the last to be divisible by q. Hence, one can forget its factors. However, one needs as many factors at the numerator than at the denominator. For the other terms, the following table of the rest of $d \cdot a$ gives the

† Associated with these various hypergeometric functions with the same singularities at $x = 0, 1, \infty$.

complete proof:

.	1	2	4	5	7	8
1	1	2	4	5	7	8
4	4	8	7	2	1	5
5	5	1	2	7	8	4
3	3	6	3	6	3	6

(G.1)

One finds out that this is always a factor of the numerator, before the occurrence of a factor at the denominator.

Appendix H. Integrality of differential geometry modular form series

Appendix H.1. Golyshev and Stienstra examples [117]

From a differential Geometry viewpoint, Golyshev and Stienstra gave a set of selected order-three linear differential operators in [117]. The Wronskians of all these Golyshev and Stienstra examples, displayed in [117], are square roots of simple rational functions. Consequently, the differential Galois groups of the order-three operators displayed in [117] will be $O(3, \mathbb{C})$ instead of $SO(3, \mathbb{C})$.

Furthermore, all these Golyshev and Stienstra order-three operators are symmetric squares of order-two linear differential operators.

For instance for G_5, it is the symmetric square of

$$\mathcal{H}_5 \quad = \quad D_x^2 \quad + \frac{1-66\,x-32\,x^2}{(1-44\,x\,-16\,x^2)\cdot x}\cdot D_x \quad - \frac{3\cdot\,(x+1)}{(1-44\,x\,-16\,x^2)\cdot x}. \qquad \text{(H.1)}$$

This Heunian second order operator $\mathcal{H}_5$ is actually solvable in ${}_2F_1$ hypergeometric function with a (modular) pullback

$$\kappa^{1/4}\,\cdot\,{}_2F_1\Big([\frac{1}{12},\frac{5}{12}],\,[1];\,P_u\Big) \qquad \text{(H.2)}$$

To see this, one can, for example, calculate the nome q in terms of the variable x from the series solutions $y_0(x)$ and $y_1(x)$ in x of (H.1), i.e. from $q = \exp(y_1(x)/y_0(x))$, then obtain a series expansion in x for P_u using the well-known expansion of the modular invariant $j(q) = 1728/P_u = 1/q + 784 + 196884q + 21493760q^2 + \cdots$ in terms of q, and then recognise the algebraic equation satisfied by P_u using for example Maple's command gfun[seriestoalgeq]. One obtains:

$$(144\,x^2+216\,x+1)^3\cdot\,P_u^2\,-1728\,x\,(3456\,x^5+7776\,x^4-12600\,x^3 \\ +1890\,x^2-80\,x+1)\cdot\,P_u\,+2985984\,x^6 \quad = \quad 0, \qquad \text{(H.3)}$$

then:

$$P_u \;=\; \frac{864\,x\cdot\,(1\,-80\,x\,+1890\,x^2\,-12600\,x^3\,+7776\,x^4\,+3456\,x^5)}{(1+216\,x+144\,x^2)^3} \qquad \text{(H.4)}$$
$$+\,\frac{864\cdot\,x\cdot\,(1-4\,x)\,(1\,-18\,x)\,(1\,-36\,x)}{(1+216\,x+144\,x^2)^3}\cdot(1\,-44\,x\,-16\,x^2)^{1/2},$$

$$\kappa \;=\; \frac{5\cdot\,(36\,x-13)\,+60\cdot\,(1\,-44\,x\,-16\,x^2)^{1/2}}{1\,+216\,x\,+144\,x^2}.$$

They correspond to the following series expansions:

$$P_u \;=\; 1728\,x \;-\; 1257984\cdot x^2 \;+\; 575828352\cdot x^3 \;-\; 214274336256\cdot x^4 \;+\; 70880897026368\cdot x^5 \;-\; 21731780729723904\cdot x^6 \;+\; \cdots,$$

$$\kappa \;=\; -125 \;+28500\,x \;-6123000\,x^2 \;+1318794000\,x^3 \;-283968657000\,x^4 \;+61147607046000\,x^5 \;-13166982207738000\,x^6 \;+\; \cdots$$

Note that the series for κ, and P_u, have *integer coefficients*.

Introducing the rational parametrisation of curve $y^2 \,-\, (1 - 44\,x - 16\,x^2) \,=\, 0$, in order to get rid of the square root $(1 - 44\,x - 16\,x^2)^{1/2}$, namely

$$x \;=\; \frac{\mu}{(125\,+22\,\mu\,+\mu^2)}, \qquad\qquad y \;=\; \pm\,\frac{(\mu^2-125)}{(125\,+22\,\mu\,+\mu^2)}, \tag{H.5}$$

the corresponding two pullbacks and κ's reading

$$P_u \;=\; \frac{1728\,\mu}{(5\,+10\,\mu\,+\mu^2)^3}, \qquad\qquad \kappa_u \;=\; -\frac{5\,(125\,+22\,\mu\,+\mu^2)}{(5\,+10\,\mu\,+\mu^2)},$$

$$P_v \;=\; \frac{1728\,\mu^5}{(3125\,+220\,\mu\,+\mu^2)^3}, \qquad\qquad \kappa_v \;=\; -\frac{125\,(125\,+22\,\mu\,+\mu^2)}{(3125\,+250\,\mu\,+\mu^2)}.$$

where it is straightforward to see the *"Atkin" symmetry* $\mu \;\longleftrightarrow\; 125/\mu$:

$$P_v(\mu) \;=\; P_u\Big(\frac{125}{\mu}\Big), \qquad\qquad \kappa_v(\mu) \;=\; \kappa_u\Big(\frac{125}{\mu}\Big), \tag{H.6}$$

$$x\Big(\frac{125}{\mu}\Big) \;=\; x(\mu), \qquad y\Big(\frac{125}{\mu}\Big) \;=\; -\,y(\mu), \qquad \lambda_5\Big(\frac{125}{\mu}\Big) \;=\; -\,\lambda_5(\mu).$$

The relation between the two pullbacks P_u and P_v corresponds to a (rational) modular curve. *One immediately recognises the modular curve* corresponding to the elimination of μ between (the two Hauptmoduls) P_u, and P_v, which is well-known to correspond to $q \,\rightarrow\, q^5$, or $\tau \,\rightarrow\, 5\cdot\tau$ (namely the fundamental modular curve $\Phi(j(\tau),\, j(5\cdot\tau)) \,=\, 0$).

All these results for equation (H.1) can be summarised in the following equation

$$\rho_5(a_5\cdot\, x)^{-1}\cdot \mathrm{Pullback}\Big[\mathcal{H}_5, x \rightarrow c_5(a_5\cdot\, x)\Big]\cdot \rho_5(a_5\cdot\, x) \quad=\quad \omega_5, \tag{H.7}$$

where we denote by ω_5, the order-two operator corresponding to the modular solution $D_5\,\cdot\,{}_2F_1\Big([\frac{1}{12},\frac{5}{12}],\,[1];\,\frac{1728}{j_5}\Big)$, where D_5 and j_5 are given by Maier (see tables 4 and 12 in [123]). The expression of $c_5(x)$ is given in (G.4): $c_5(x) \,=\, x/(x^2\,+\,22\,x\,+\,125)$ and $\rho_5(x)$ is given below.

This means that the order-two linear differential operator, associated with Golyshev and Stienstra example G_5, when pull-backed by appropriate functions $c_5(x)$, and $\rho_5(x)$, is simply the operator corresponding to the modular solution j_5.

In a similar way, one can perform the same calculations for other examples given in [144]. The results are displayed in the following table with the same notations as before:

$$\rho_n(a_n\cdot\, x)^{-1}\cdot \mathrm{Pullback}\Big[\mathcal{H}_n, x \rightarrow c_n(a_n\cdot\, x)\Big]\cdot\rho_n(a_n\cdot\, x) \quad=\quad \omega_n, \tag{H.8}$$

n	$c_n(x)$	$\rho_n(x)$	$a_n = A_n \cdot B_n$
2	$x/(x+64)^2$	$(x+64)^{1/4}$	2^{12}
3	$x/(x+27)^2$	$(x+27)^{1/3}$	3^6
4	$x/(x+16)^2$	$(x+16)^{1/2}$	2^8
5	$x/(x^2+22x+125)$	$(x^2+22x+125)^{1/4}$	$2^2\, 5^3$
6	$x/(x+9)/(x+8)$	$((x+9)(x+8))^{1/2}$	$2^3\, 3^2$
7	$x/(x^2+13x+49)$	$(x^2+13x+49)^{1/3}$	$7^2\, 3^2$
8	$x/(x+8)/(x+4)$	$((x+8)(x+4))^{1/2}$	2^5
9	$x/(x^2+9x+27)$	$(x^2+9x+27)^{1/2}$	3^3

Operators $\mathcal{H}_n$ for $n = 6, 7, 8, 9$ are given in the next subsection (for $n = 2, 3, 4$, see [144]).

Let us note that all these examples are associated to genus zero curves. Another example G_{11} given in [144] which is associated to a genus one curve will be considered in detail in Appendix I.

Appendix H.2. More details

The order-three operator G_6 is the *symmetric square* of

$$\mathcal{H}_6 \quad = \quad D_x^2 \quad + \frac{1-51\,x+2\,x^2}{x\cdot(1-34\,x+x^2)}\cdot D_x \quad + \quad \frac{x-10}{4\,x\cdot(1-34\,x+x^2)}. \qquad \text{(H.9)}$$

Let us introduce the rational parametrisation of $y^2 - (1-34\,x+x^2) = 0$:

$$x \;=\; \frac{\mu}{(\mu+9)\cdot(\mu+8)}, \qquad\qquad y \;=\; \frac{\mu^2-72}{(\mu+9)\cdot(\mu+8)}.$$

With this new parametrisation the operator (H.9) becomes

$$L_\mu \quad = \quad 4\cdot(\mu+9)^2\cdot(\mu+8)^2\cdot\theta_\mu^2 \quad -10\cdot(\mu+9)\cdot(\mu+8)\cdot\mu \quad +\mu^2, \qquad \text{(H.10)}$$

which is covariant by the “Atkin” involution $\mu \leftrightarrow 72/\mu$, x being invariant by this involution:

$$x\Bigl(\frac{72}{\mu}\Bigr) \;=\; x(\mu), \qquad\qquad y\Bigl(\frac{72}{\mu}\Bigr) \;=\; -\,y(\mu), \qquad \text{(H.11)}$$

It is worth recalling the rational parametrisation of the modular curve $\tau \rightarrow 6\cdot\tau$ namely [123]:

$$j_6 \;=\; \frac{(\mu+6)^3\cdot(\mu^3+18\,\mu^2+84\,\mu+24)}{\mu\cdot(\mu+8)^3\cdot(\mu+9)^2}, \qquad \text{(H.12)}$$

$$j_6' \;=\; \frac{(\mu+12)^3\cdot(\mu^3+252\,\mu^2+3888\,\mu+15552)}{\mu^6\cdot(\mu+8)^2\cdot(\mu+9)^3} \;=\; j_6\Bigl(\frac{72}{\mu}\Bigr).$$

The solutions of (H.10) read:

$$\Bigl(\frac{(\mu+8)^3\,(\mu+9)^4}{\mu}\Bigr)^{1/12}\cdot {}_2F_1\Bigl([\frac{1}{12},\frac{1}{12}],\,[\frac{2}{3}];\,\frac{j_6}{1728}\Bigr), \qquad \text{(H.13)}$$

but can also be written as:

$$\Bigl(\frac{(\mu+8)^2\,(\mu+9)^2}{(\mu+6)\,(\mu^3+18\,\mu^2+84\,\mu+24)}\Bigr)^{1/4}\cdot {}_2F_1\Bigl([\frac{1}{12},\frac{5}{12}],\,[1];\,\frac{1728}{j_6}\Bigr), \qquad \text{(H.14)}$$

$$\Bigl(\frac{36\cdot(x+8)^2\,(x+9)^2}{(\mu+12)\cdot(\mu^3+252\,\mu^2+3888\,\mu+15552)}\Bigr)^{1/4}\cdot {}_2F_1\Bigl([\frac{1}{12},\frac{5}{12}],\,[1];\,\frac{1728}{j_6'}\Bigr).$$

The third order operator G_7 is the symmetric square of the second order linear differential operator

$$\mathcal{H}_7 \, = \, D_x^2 \, + \frac{1-39\,x-54\,x^2}{(1-27\,x)\,(x+1)\,x} \cdot D_x \, - \, \frac{2\cdot(1+3\,x)}{(1-27\,x)\,(x+1)\,x}. \qquad (H.15)$$

Introducing the parametrisation of the rational curve

$$y^2 \, - \, (1\,+x)\cdot(1-27\,x) \, = \, 0, \qquad (H.16)$$

namely

$$x \, = \, \frac{\mu}{\mu^2\,+13\,\mu\,+49}, \qquad y \, = \, \frac{\mu^2\,-49}{\mu^2\,+13\,\mu\,+49}, \qquad (H.17)$$

where one verifies the existence of an "Atkin" involution:

$$x\Bigl(\frac{49}{\mu}\Bigr) \, = \, x(\mu), \qquad y\Bigl(\frac{49}{\mu}\Bigr) \, = \, -\,y(\mu), \qquad (H.18)$$

With this change of variables the second order differential operator (H.15) reads:

$$L_\mu \, = \, (\mu^2+13\,\mu\,+49)^2\cdot\theta_\mu^2 \, -2\cdot(\mu^2+13\,\mu\,+49)\cdot\mu \, -6\,\mu^2. \qquad (H.19)$$

It is worth recalling the rational parametrisation of the modular curve $\tau \, \rightarrow \, 7\cdot\tau$ namely [123]:

$$j_7 \, = \, \frac{(\mu^2\,+13\,\mu\,+49)\cdot(\mu^2\,+5\,\mu\,+\,1)}{\mu}, \qquad (H.20)$$

$$j_7' \, = \, \frac{(\mu^2\,+13\,\mu\,+49)\cdot(\mu^2\,+245\,\mu\,+\,2401)}{\mu^7} \, = \, j_7\Bigl(\frac{49}{\mu}\Bigr).$$

The solution of (H.19) reads:

$$\Bigl(\frac{(\mu^2\,+13\,\mu\,+49)^4}{\mu}\Bigr)^{1/12} \cdot {}_2F_1\Bigl([\frac{1}{12},\frac{1}{12}],\,[\frac{2}{3}];\,\frac{j_7}{1728}\Bigr),$$

$$\Bigl(\frac{7^6\cdot(\mu^2\,+13\,\mu\,+49)^4}{\mu^7}\Bigr)^{1/12} \cdot {}_2F_1\Bigl([\frac{1}{12},\frac{1}{12}],\,[\frac{2}{3}];\,\frac{j_7'}{1728}\Bigr).$$

The order-three operator G_8 is the symmetric square of

$$\mathcal{H}_8 \, = \, D_x^2 \, + \frac{1\,-36\,x\,+\,32\,x^2}{x\cdot(1\,-24\,x\,+\,16\,x^2)} \cdot D_x \, - \, 2\,\frac{1\,-2\,x}{x\cdot(1\,-24\,x\,+\,16\,x^2)}. \qquad (H.21)$$

Furthermore one has an "Atkin" symmetry $x \, \leftrightarrow \, 1/16/x$. Introducing the parametrisation of $y^2 \, - \, (1\,-24\,x\,+16\,x^2) \, = \, 0$, namely

$$x \, = \, \frac{\mu}{(\mu\,+4)\cdot(\mu\,+8)}, \qquad y \, = \, \frac{\mu^2\,-32}{(\mu\,+4)\cdot(\mu\,+8)}, \qquad (H.22)$$

the linear differential operator (H.21) becomes

$$L_\mu \, = \, (\mu\,+4)^2\cdot(\mu\,+8)^2\cdot\theta_\mu^2 \, -2\cdot(\mu\,+4)\cdot(\mu\,+8)\cdot\mu \, +4\cdot\mu^2,$$

this operator being covariant by the "Atkin" involution which leaves x invariant:

$$x\Bigl(\frac{32}{\mu}\Bigr) \, = \, x(\mu), \qquad y\Bigl(\frac{32}{\mu}\Bigr) \, = \, -\,y(\mu). \qquad (H.23)$$

It is worth recalling the rational parametrisation of the modular curve representing $\tau \rightarrow 8 \cdot \tau$, namely [123]:

$$j_8 \, = \, \frac{(\mu^4 \, + \, 16\,\mu^3 \, + 80\,\mu^2 \, + 128\,\mu \, + 16)^3}{\mu \cdot (\mu \, + 4)^2 \, (\mu \, + 8)}, \tag{H.24}$$

$$j'_8 \, = \, \frac{(\mu^4 \, + \, 256\,\mu^3 \, + 5120\,\mu^2 \, + 32768\,\mu \, + 65536)^3}{\mu \cdot (\mu \, + 4) \, (\mu \, + 8)^2} \, = \, j_8\Big(\frac{32}{\mu}\Big).$$

A solution of (H.23) reads

$$\Big(\frac{(\mu \, + \, 8)^5 (\mu \, + \, 4)^4}{\mu}\Big)^{1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{1}{12}], \, [\frac{2}{3}]; \, \frac{j_8}{1728}\Big), \tag{H.25}$$

but can also be written

$$\Big((\mu \, + 4) \cdot (\mu \, + 8)\Big)^{1/2} \cdot {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], \, [1]; \, -\frac{\mu \cdot (\mu \, + 8)}{16}\Big),$$

or equivalently, using Gauss-Kummer identity:

$$\Big((\mu \, + 4) \cdot (\mu \, + 8)\Big)^{1/2} \cdot {}_2F_1\Big([\frac{1}{4}, \frac{1}{4}], \, [1]; \, -\frac{\mu \cdot (\mu \, + 8) \cdot (\mu \, + 4)^2}{64}\Big).$$

Finally, the order-three operator G_9 is the symmetric square of

$$\mathcal{H}_9 \, = \, D_x^2 \, + \, \frac{1 \, - 27\,x \, - \, 54\,x^2}{x \cdot (1 \, - 18\,x \, - \, 27\,x^2)} \cdot D_x \, - \, \frac{6 \, + 27\,x}{4\,x \cdot (1 \, - 18\,x \, - \, 27\,x^2)}. \tag{H.26}$$

With the parametrisation of the rational curve $y^2 \, - \, (1 \, - 18\,x \, - \, 27\,x^2) \, = \, 0$, namely

$$x \, = \, \frac{\mu}{\mu^2 \, + 9\,\mu \, + 27}, \qquad \qquad y \, = \, \frac{\mu^2 \, - 27}{\mu^2 \, + 9\,\mu \, + 27}, \tag{H.27}$$

the linear differential operator (H.26) becomes

$$L_\mu \, = \, 4 \cdot (\mu^2 \, + 9\,\mu \, + 27)^2 \cdot \theta_\mu^2 \, - 6 \cdot (\mu^2 \, + 9\,\mu \, + 27) \cdot \mu \, - 27\,\mu^2,$$

which is covariant by the "Atkin" involution leaving x invariant:

$$x\Big(\frac{27}{\mu}\Big) \, = \, x(\mu), \qquad \qquad y\Big(\frac{27}{\mu}\Big) \, = \, -\,y(\mu). \tag{H.28}$$

It is worth recalling the rational parametrisation of the modular curve $\tau \rightarrow 9 \cdot \tau$ namely [123]:

$$j_9 \, = \, \frac{(\mu \, + 3)^3 \cdot (\mu^3 \, + 9\,\mu^2 + \, 27\,\mu \, + \, 3)}{\mu \cdot (\mu^2 \, + 9\,\mu \, + 27)}, \tag{H.29}$$

$$j'_9 \, = \, \frac{(\mu \, + 9)^3 \cdot (\mu^3 \, + 243\,\mu^2 + \, 2187\,\mu \, + \, 6561)}{\mu^9 \cdot (\mu^2 \, + 9\,\mu \, + 27)} \, = \, j_9\Big(\frac{27}{\mu}\Big).$$

A solution of (H.28) reads

$$\Big(\frac{(\mu^2 \, + \, 9\,\mu \, + 27)^5}{\mu}\Big)^{1/12} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{1}{12}], \, [\frac{2}{3}]; \, \frac{j_9}{1728}\Big), \tag{H.30}$$

but can also be written as:

$$\Big((\mu^2 \, + \, 9\,\mu \, + 27)\Big)^{1/2} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{1}{3}], \, [1]; \, -\frac{\mu \cdot (\mu^2 \, + \, 9\,\mu \, + 27)}{27}\Big), \tag{H.31}$$

$$\Big(\frac{27 \cdot (\mu^2 \, + \, 9\,\mu \, + 27)}{\mu^2}\Big)^{1/2} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{1}{3}], \, [1]; \, -27\,\frac{\mu^2 \, + \, 9\,\mu \, + 27}{\mu^3}\Big).$$

The relation between these two pullbacks corresponds to the modular curve

$$x^3y^3 \, - 270 \cdot x^2y^2 \, + 972 \cdot x\,y \cdot (x \, + y) \, - 729 \cdot (x^2 \, + \, xy \, + \, y^2) \, = \, 0. \tag{H.32}$$

Appendix I. Higher genus modular forms

For *non-zero genus* modular curves, we have generalisations of these structures associated with an "Atkin" involution of the form $z \rightarrow A/z$, which correspond to the introduction of the so-called *Atkin*† *modular polynomials, or star-modular polynomials* [146].

Appendix I.1. A genus-one curve

Let us first consider the order-three Golyshev and Stienstra linear differential operator G_{11} given in [144] which is the symmetric square of

$$\mathcal{H}_{11} \,\, = \,\, D_x^2 \,\, + \frac{(625 - 12750\,x - 30800\,x^2 - 3150\,x^3 - 4512\,x^4)}{x \cdot \, (125 \, - 1900\,x \, - 40\,x^2 \, - 188\,x^3) \cdot \, (8\,x+5)} \cdot D_x \\ - \frac{6 \cdot \, (125 + 700\,x + 105\,x^2 + 188\,x^3)}{(125 \, - 1900\,x \, - 40\,x^2 \, - 188\,x^3) \cdot \, (8\,x+5)}. \qquad \text{(I.1)}$$

Performing the same calculations as for (H.1), one deduces that the solution of (I.1) can be written in terms of pull-backed hypergeometric function, namely ${}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1]; \, P_u\Big)$, with $P_u \, = \, 1728/J$, with J satisfying the following algebraic equation:

$$244140625\,x^{12} \cdot \, J^2 \, - 5 \cdot \, (132645814272\,x^{11} \, + 372815032320\,x^{10} \, + 1405869696000\,x^9 \\ + 5229172080000\,x^8 \, - 225383400000\,x^7 \, - 5599578600000\,x^6 \, + 339591656250\,x^5 \\ + 1103850000000\,x^4 \, - 349421875000\,x^3 \, + 42861328125\,x^2 \, - 2363281250\,x \\ + 48828125) \cdot \, x \cdot \, J \, + (78336\,x^4 + 181440\,x^3 + 561600\,x^2 + 144000\,x + 625)^3 \,\, = \,\, 0.$$

Then, when we perform the pullback $x \,\rightarrow\, 5/(5\,z - 3)$, we obtain the modular polynomial of order-eleven given† by Morain [146], or by Elkies (see eq. (49) in [147]), associated to a *genus-one* modular curve:

$$J^2 \,\, - Q_1(z) \cdot \, J \,\, + (z^4 + 228\,z^3 + 486\,z^2 - 540\,z + 225)^3 \,\, = \,\, 0. \qquad \text{(I.2)}$$

where:

$$Q_1(z) \,\, = \,\, z^{11} \,\, - 55\,z^{10} \, + 1188\,z^9 \, - 12716\,z^8 \, + 69630\,z^7 \, - 177408\,z^6 \\ + 133056\,z^5 \,\, + 132066\,z^4 \,\, - 187407\,z^3 \,\, + 40095\,z^2 \,\, + 24300\,z \,\, - 6750.$$

The j-invariant of the genus-one $(J, \, z)$-curve (I.2) reads¶:

$$j_{inv} \,\, = \,\, -\frac{122023936}{161051} \,\, = \,\, -\frac{496^3}{11^5}. \qquad \text{(I.3)}$$

Appendix I.1.1. Atkin modular polynomial and modular forms for $\tau \,\rightarrow\, 11\,\tau$

Let us focus on the previous *Atkin modular polynomial* (I.2):

$$\Phi^*_{11}(z, \, j) \,\, = \,\, j^2 \,\, - Q_1(z) \cdot j \,\, + \,\, (z^4 + 228\,z^3 + 486\,z^2 - 540\,z + 225)^3. \qquad \text{(I.4)}$$

Note that there is no associated *"Atkin" involution* of the (rational) form $z \,\rightarrow\, A/z$, since the curve $\Phi^*_{11}(z, \, j) \, = \, 0$ is a *genus-one* curve.

† Atkin's work we are interested in, cannot be found in papers but in emails [145].
† See $\Phi^*_{11}(F, \, J)$ in the last equation of subsection 2.3.2 of [146]. The variable F in [146] is z here.
¶ Use the command algcurves[j_invariant] in Maple.

The elimination of z, between the two solutions j_+ and j_- of $\Phi_{11}^*(z, j) = 0$, *yields the modular curve*

$$\Phi_{11}(j_+, j_-) \quad = \quad 0, \tag{I.5}$$

which is a quite large (146 monomials) (j_+, j_-)-symmetric polynomial of degree twelve in j_+ (resp. j_-). The discriminant of $\Phi_{11}(j_+, j_-)$ in j_- reads (we denote here j_- by j):

$$- 11^{11} \cdot j^8 \cdot (j - (12)^3)^6 \cdot \mathcal{Q}^2 \cdot \mathcal{P}^2, \tag{I.6}$$

where $\mathcal{Q}$ reads:

$$\mathcal{Q} \quad = \quad (j + (15)^3) \cdot (j - (20)^3) \cdot (j + (96)^3) \cdot (j + (960)^3) \cdot (j - (255)^3)$$
$$\times \, (j^2 - 425692800\, j + 9103145472000)^2 \cdot (j^2 + 117964800\, j - 134217728000)^2,$$

and where $\mathcal{P}$ is a polynomial that factors into the product of seven polynomials of degree two, fourteen polynomials of degree four, one polynomial of degree six and four polynomials of degree eight:

$$\mathcal{P} \quad = \quad P_2^{(1)} \, \cdots \, P_2^{(7)} \cdot P_4^{(1)} \, \cdots \, P_4^{(14)} \cdot P_6^{(1)} \cdot P_8^{(1)} \, \cdots \, P_8^{(4)}. \tag{I.7}$$

In the $j_+ = j_- = j$ limit it becomes a polynomial of degree 22, which simply factors as follows:

$$\Phi_{11}(j, j) \quad = \quad - (j + (32)^3) \cdot \mathcal{Q}^2$$
$$\times \, (j^3 - 1122662608\, j^2 + 270413882112\, j - 653249011576832).$$

where, besides the *Complex Multiplication* value [148] $j = (255)^3 = 16581375$, one recognises a large set of *Heegner numbers* [17], corresponding to the following integer values of the j-invariant: $(20)^3$, $-(15)^3$, $-(32)^3$, $-(96)^3$, $-(960)^3$. Note that the j-invariant of the genus-one (j_+, j_-) modular curve also reads the *same j-invariant* as the one for the genus-one curve (I.2), namely $-122023936/161051$ (see (I.3)).

Recalling the well-known expansion of the j-function as a function of q:

$$j(q) \; = \; \frac{1}{q} \; + 744 \; + 196884\, q \; + 21493760\, q^2 \; + 864299970\, q^3 \tag{I.8}$$
$$+ 20245856256\, q^4 + 333202640600\, q^5 + 4252023300096\, q^6 \; + \; \cdots$$

one verifies immediately that *it does correspond* to $\tau \, \rightarrow \, 11\,\tau$:

$$\Phi_{11}\Big(j(q), \, j(q^{11})\Big) \quad = \quad 0. \tag{I.9}$$

Recalling the modular discriminant of Weierstrass, $\Delta(q)$ (i.e. the 24-th power of the Dedekind eta function up to $(2\pi)^{12}$ factor), it reads:

$$\Delta(q) \quad = \quad q \cdot \prod_{n=1}^{\infty} (1 \, - \, q^n)^{24}.$$

Let us introduce the *Eisenstein series*‡ $E_2(q)$ and the Euler-like function $\rho(q)$:

$$E_2(q) \; = \; \frac{q}{24} \cdot \frac{d \ln(G(q))}{d\, q}, \qquad \text{where:} \qquad G(q) \; = \; \frac{\Delta(q^{11})}{\Delta(q)},$$

$$\rho(q) \quad = \quad q \cdot \prod_{n=1}^{\infty} (1 \, - \, q^n)^2 \cdot \prod_{n=1}^{\infty} (1 \, - \, q^{11\,n})^2. \tag{I.10}$$

‡ Quasi-modular form [149]: $G_2((a\,\tau + b)/(c\,\tau + d)) \, = \, (c\,\tau + d)^2 \cdot G_2(\tau) \, - \, (c\,\tau + d)\, c/4/\pi/i$.

and, finally $z(q)$ (denoted F in Morain [146]):

$$z(q) \;=\; \frac{3}{5} + \frac{12}{5}\cdot\frac{E_2(q)}{\rho(q)} \;=\; \frac{1}{q} \;+5\; +17\,q\; +46\,q^2 +116\,q^3 \;+252\,q^4 \;+533\,q^5 \;+1034\,q^6 \;+1961\,q^7 \;+\;\cdots \qquad (I.11)$$

One verifies that the two relations on the *Atkin modular polynomial* (I.4) are satisfied:

$$\Phi^*_{11}(z(q),\, j(q)) \;=\; 0, \qquad \text{and:} \qquad \Phi^*_{11}(z(q),\, j(q^{11})) \;=\; 0. \qquad (I.12)$$

Introducing $x(q) \;=\; 1/z(q)$ one has the following series expansion with integer coefficients:

$$x(q) \;=\; q \;-5\,q^2 \;+8\,q^3 - q^4 - 17\,q^5 + 62\,q^6 - 176\,q^7 + 339\,q^8 - 386\,q^9 \;+\;\cdots$$

Let us now show an identity characteristic of modular forms:

$$A(x)\cdot\,{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1];\,\frac{1728}{j(q^{11})}\Big) \;=\; {}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1];\,\frac{1728}{j(q)}\Big), \qquad (I.13)$$

where $A(x)$ is the algebraic function

$$\frac{A(x)^4}{11^2} + \frac{11^2}{A(x)^4} \;=\; \frac{2}{11^2}\cdot\frac{7321 - 87612\,x + 73206\,x^2 + 21060\,x^3 - 23175\,x^4}{1 + 228\,x + 486\,x^2 - 540\,x^3 + 225\,x^4}.$$

Appendix I.1.2. A change of variables

Let us rewrite (I.4) in this variable $x \;=\; 1/z$, and in the Hauptmodul $H \;=\; 1728/j$, introducing a new star-modular polynomial $P^*_{11}(x,\,H) = H^2\cdot z^{12}\cdot\Phi^*_{11}(1/x,\,1728/H)$:

$$P^*_{11}(x,\,H) \;=\; (1\,+228\,x\,+486\,x^2\,-540\,x^3 + 225\,x^4)^3\cdot\,H^2 \;-1728\cdot\,\mathcal{Q}_1(x)\cdot x\,\cdot\,H \;+1728^2\,x^{12}, \qquad \text{with:} \qquad (I.14)$$

$$\mathcal{Q}_1(x) \;=\; 1 \;-55\,x \;+1188\,x^2 \;-12716\,x^3 \;+69630\,x^4 \;-177408\,x^5 + 133056\,x^6 \;+132066\,x^7 - 187407\,x^8 + 40095\,x^9 + 24300\,x^{10} - 6750\,x^{11}.$$

The two Hauptmodul solutions of polynomial $P^*_{11}(x,\,H)$ expand respectively as:

$$\frac{1}{j(q)} \;=\; \frac{H_1}{1728} \;=\; x \;-739\,x^2 \;+349254\,x^3 - 135092042\,x^4 + 46600204623\,x^5 \;+\;\cdots,$$

$$\frac{1}{j(q^{11})} \;=\; \frac{H_2}{1728} \;=\; x^{11} \;+55\,x^{12} + 1837\,x^{13} + 48411\,x^{14} + 1109999\,x^{15} \;+23244727\,x^{16} \;+\;\cdots \qquad (I.15)$$

The corresponding expansions of ${}_2F_1([1/12,\,5/12],\,[1];\,H_1)$ and ${}_2F_1([1/12,\,5/12],\,[1];\,H_2)$ read respectively two series with *integer* coefficients:

$${}_2F_1([1/12,\,5/12],\,[1];\,H_1) \;=\; 1 \;+60\,x - 4560\,x^2 + 614400\,x^3 - 95660400\,x^4 \;+16231863060\,x^5 - 2905028387700\,x^6 \;+\;\cdots\; + 20000242239261022140\,x^9 \;-3953288123422938241560\,x^{10} +\; 791518845663517087144740\,x^{11} \;+\;\cdots$$

$${}_2F_1([1/12,\,5/12],\,[1];\,H_2) \;=\; 1 \;+60\,x^{11} \;+3300\,x^{12} \;+110220\,x^{13} \;+2904660\,x^{14} \;+66599940\,x^{15} \;+1394683620\,x^{16} \;+27425371380\,x^{17} \;+\;\cdots \qquad (I.16)$$

Defining $A(x)$ as the ratio of these two series expansions:

$${}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1];\,H_1\Big) \;=\; A(x)\cdot\,{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1];\,H_2\Big), \qquad (I.17)$$

one can easily see that this ratio $A(x)$ is solution of the *genus-one* algebraic equation:

$$\frac{A(x)^4}{11^2} + \frac{11^2}{A(x)^4} = \frac{2}{11^2} \cdot \frac{\gamma(x)}{\delta(x)}. \tag{I.18}$$

It is worth noting that this *genus-one* algebraic curve (I.18), in the two variables x and $y = A(x)^4$, has the *same* j*-invariant*, namely $-122023936/161051$ (see (I.3)), as the *genus-one* (x, y)-curve (I.2) corresponding to the *Atkin-modular polynomial*.

The ratio $A(x)$ then reads:

$$A(x)^4 = \frac{\gamma(x) - 120 \cdot \alpha(x) \cdot \beta(x)^{1/2}}{\delta(x)}, \qquad \text{where:} \tag{I.19}$$

$$\alpha(x) = 45\,x^2 - 246\,x + 61, \qquad \beta(x) = (1+x) \cdot (1 - 17\,x + 19\,x^2 - 7\,x^3),$$
$$\gamma(x) = 7321 - 87612\,x + 73206\,x^2 + 21060\,x^3 - 23175\,x^4,$$
$$\delta(x) = 1 + 228\,x + 486\,x^2 - 540\,x^3 + 225\,x^4. \tag{I.20}$$

Its series expansion is a series with integer coefficients:

$$A(x) = 1 + 60\,x - 4560\,x^2 + 614400\,x^3 - 95660400\,x^4 + 16231863060\,x^5$$
$$- 2905028387700\,x^6 + \cdots + 20000242239261022140\,x^9 \tag{I.21}$$
$$- 3953288123422938241560\,x^{10} + 791518845663517087144680\,x^{11} + \cdots$$

Introducing the polynomial $\zeta(x) = 1 + 8\,x - 9\,x^2 + 10\,x^3 - 6\,x^4$, one has the following relations on $A(x)$:

$$\frac{d \ln A(x)}{dx} = \frac{60 \cdot \zeta(x)}{\beta(x)^{1/2} \cdot \delta(x)}, \tag{I.22}$$

Appendix I.1.3. Order-four operator

Let us introduce the order-four linear differential operator L_4 annihilating ${}_2F_1\Big([1/12, 5/12], [1]; H_1\Big)$. Quite remarkably this linear differential operator L_4 *also annihilates* ${}_2F_1\Big([1/12, 5/12], [1]; H_2\Big)$. Therefore, this order-four operator is not a MUM operator (it has two series-solutions (I.16), analytic at $x = 0$).

The symmetric square of L_4 is of *order nine*, instead of the order ten one can expect generically. Furthermore, the exterior square of L_4 is of order six, but, remarkably, this exterior square factorizes into a *direct sum of an order-one operator*, M_1, *an order-two operator*, $M_2^{(1)}$, and an *order-three operator*, which is the symmetric square of *another order-two operator*, $M_2^{(2)}$:

$$ext^2(L_4) = M_1 \oplus M_2^{(1)} \oplus Sym^2(M_2^{(2)}), \tag{I.23}$$

where

$$M_1 = D_x - \frac{1}{4} \cdot \frac{d \ln(\mu(x))}{dx}, \qquad \text{with:} \qquad \mu(x) = \frac{\delta(x)}{x^4 \cdot \beta(x)^2} \tag{I.24}$$

the Wronskian of the two order-two operators $M_2^{(1)}$ and $M_2^{(2)}$ reading respectively:

$$Wr(M_2^{(1)})^2 = \frac{\zeta(x)^2}{x^4 \cdot \beta(x)^3 \cdot \delta(x)}, \qquad Wr(M_2^{(2)})^4 = \frac{\zeta(x)^4}{x^4 \cdot \beta(x) \cdot \delta(x)^3}. \tag{I.25}$$

This factorisation is a consequence of relation (I.17) and indicates that L_4 is very special.

The Hauptmoduls H_1 or H_2 are *not rational functions* of x, they are *algebraic*. Therefore ${}_2F_1\Big([1/12, 5/12], [1]; H_1\Big)$ has no reason to be solution of a second-order operator. As far as factorisation of linear differential operators in operators with polynomial coefficients, L_4 is irreducible†. Let us show that such a reduction to a second order linear differential operator actually exists.

Appendix I.1.4. Order-two operator: operator ω_{11}

Introducing the algebraic function $\mathcal{A}(x)$

$$\mathcal{A}(x) \quad = \quad \frac{1}{\delta(x)^{1/8} \cdot A(x)^{1/2}} \quad = \quad \Big(\frac{1}{\gamma(x) \, - \, 120 \cdot \alpha(x) \cdot \beta(x)^{1/2}}\Big)^{1/8}$$
$$= \quad \frac{\beta(x)^{1/4} \cdot \delta(x)^{3/8}}{\zeta(x)^{1/2}} \cdot \Big(\frac{-1}{60} \cdot \frac{d\,(1/A(x))}{dx}\Big)^{1/2}, \qquad \text{(I.26)}$$

which has the series expansion

$$\mathcal{A}(x) \quad = \quad 1 \quad - \frac{117}{2}\,x \ + \frac{64635}{8}\,x^2 \ - \frac{21853425}{16}\,x^3 \ + \frac{32050683795}{128}\,x^4$$
$$- \frac{12299248285371}{256}\,x^5 \ + \frac{9718868161850799}{1024}\,x^6 \quad + \ \cdots, \qquad \text{(I.27)}$$

one finds that

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1]; H_1\Big) \quad = \quad 1 \quad + \frac{3}{2}\,x \ + \frac{75}{8}\,x^2 \ + \frac{1335}{16}\,x^3$$
$$+ \frac{111795}{128}\,x^4 \ + \frac{2559789}{256}\,x^5 \ + \frac{124177119}{1024}\,x^6 \quad + \ \cdots \qquad \text{(I.28)}$$

is actually solution of an *order-two* linear differential operator:

$$\tilde{\omega}_{11} \quad = \quad D_x^2 \quad + \frac{1 - 24\,x + 4\,x^2 + 30\,x^3 - 21\,x^4}{x\,(x+1)\,(1\, - 17\,x \, + 19\,x^2 \, - 7\,x^3)} \cdot D_x$$
$$- \frac{3\,\,(x-1)\,(7\,x^2 - x - 2)}{4\,x \cdot \,(x+1)\,(1\, - 17\,x \, + 19\,x^2 \, - 7\,x^3)}. \qquad \text{(I.29)}$$

Note that

$$M_2^{(2)} \quad = \quad \frac{1}{\lambda(x)} \cdot \tilde{\omega}_{11} \cdot \lambda(x), \quad \text{with:} \quad \lambda(x) \quad = \quad \frac{\delta(x)^{3/8} \cdot \beta(x)^{1/4}}{\zeta(x)^{1/2}}. \qquad \text{(I.30)}$$

The series (I.28) is *globally bounded*, the rescaling $x \, \rightarrow \, 4\,x$ changing this series into a series with *integer* coefficients,

$$1 \quad + 6\,x \ + 150\,x^2 \, + 5340\,x^3 \ + 223590\,x^4 \ + 10239156\,x^5 \ + 496708476\,x^6$$
$$+ 25083657720\,x^7 \ + 1304819854470\,x^8 \quad + \ \cdots, \qquad \text{(I.31)}$$

solution of the *order-two* operator (pullback of (I.29) by $x \, \rightarrow \, 4\,x$):

$$\omega_{11} \quad = \quad D_x^2 \quad + \frac{1 - 96\,x + 64\,x^2 + 1920\,x^3 - 5376\,x^4}{x \, \cdot \, (1+4\,x) \, \cdot \, (1 - 68\,x + 304\,x^2 - 448\,x^3)} \cdot D_x$$
$$- \frac{6\,(1 - 4\,x)\,(1 + 2\,x - 56\,x^2)}{x \, \cdot \, (1+4\,x) \, \cdot \, (1 - 68\,x + 304\,x^2 - 448\,x^3)}. \qquad \text{(I.32)}$$

† The command DEtools[DFactor] in Maple.

Appendix I.1.5. From order-two operator to order-four operator

This result can be revisited as follows. The function ${}_2F_1\Big([1/12, 5/12], [1]; H_1\Big)$, known to be solution of the order-four operator L_4 is also solution of the order-two operator:

$$\frac{1}{\mathcal{A}(x)} \cdot \tilde{\omega}_{11} \cdot \mathcal{A}(x) \; = \; A(x)^{1/2} \cdot \hat{\omega}_{11} \cdot \frac{1}{A(x)^{1/2}} \; = \; \hat{\omega}_{11} \, + \, \frac{1}{4} \cdot \Big(\frac{d \ln A(x)}{dx}\Big)^2 \, - \, \frac{\Omega_1}{2},$$

$$\text{where:} \qquad \hat{\omega}_{11} \; = \; \delta(x)^{1/8} \cdot \tilde{\omega}_{11} \cdot \frac{1}{\delta(x)^{1/8}} \qquad \text{(I.33)}$$

$$= \; D_x^2 \; + \; \frac{p_1(x)}{x \cdot \delta(x) \cdot \beta(x)} \cdot D_x \; + \; \frac{p_0(x)}{x \cdot \delta(x)^2 \cdot \beta(x)}$$

$$= \; D_x^2 \; - \; \frac{1}{4} \cdot \frac{d \ln(\mu(x))}{dx} \cdot D_x \; + \; \frac{p_0(x)}{x \cdot \delta(x)^2 \cdot \beta(x)},$$

with $\mu(x)$ is given in (I.24) and

$$p_1(x) \; = \; 1 \, + 147\, x \, - 4313\, x^2 \, - 7083\, x^3 \, + 14073\, x^4 \, + 4125\, x^5 \, - 19395\, x^6 \, + 12555\, x^7 \, - 3150\, x^8, \qquad \text{(I.34)}$$

$$p_0(x) \; = \; 1 \, - 19\, x \, + 1389\, x^2 \, - 6497\, x^3 \, + 27603\, x^4 \, - 37155\, x^5 \, - 18369\, x^6 \, + 45477\, x^7 \, - 17280\, x^8 \, - 270\, x^9, \qquad \text{(I.35)}$$

and where the order-one operator Ω_1 reads (using (I.22)):

$$\Omega_1 \; = \; \frac{d \ln A(x)}{dx} \cdot \Big(2\, D_x \, + \, \frac{p_1(x)}{x \cdot \delta(x) \cdot \beta(x)}\Big) \; + \; \frac{d^2 \ln A(x)}{dx^2} \qquad \text{(I.36)}$$

$$= \; \frac{60 \cdot \beta(x)^{1/2} \cdot \zeta(x)}{\beta(x) \cdot \delta(x)} \cdot \Big(2\, D_x \, - \, \frac{1}{4} \cdot \frac{d \ln(\kappa(x))}{dx}\Big), \quad \text{where:} \quad \kappa(x) \; = \; \frac{\delta(x)^5}{x^4 \cdot \zeta(x)^4}.$$

The function ${}_2F_1\Big([1/12, 5/12], [1]; H_2\Big)$, known to be solution of the order-four operator L_4 is also solution of the order-two operator:

$$\frac{1}{A(x) \cdot \mathcal{A}(x)} \cdot \tilde{\omega}_{11} \cdot \mathcal{A}(x) \cdot A(x) \; = \; \frac{1}{A(x)^{1/2}} \cdot \hat{\omega}_{11} \cdot A(x)^{1/2}$$

$$= \; \hat{\omega}_{11} \; + \; \frac{1}{4} \cdot \Big(\frac{d \ln A(x)}{dx}\Big)^2 \; + \; \frac{\Omega_1}{2}. \qquad \text{(I.37)}$$

These last two order-two operators (I.33) and (I.37) can be written as $\Omega_{11} \pm \Omega_1/2$ where:

$$\Omega_{11} \; = \; \hat{\omega}_{11} \; + \; \frac{1}{4} \cdot \Big(\frac{d \ln A(x)}{dx}\Big)^2$$

$$= \; D_x^2 \; + \; \frac{p_1(x)}{x \cdot \delta(x) \cdot \beta(x)} \cdot D_x \; - \; 30 \cdot \frac{q_0(x)}{x \cdot \delta(x)^2 \cdot \beta(x)},$$

with

$$q_0(x) \; = \; 1 \; - 49\, x \; + 909\, x^2 \, - 7877\, x^3 \, + 31323\, x^4 \, - 44025\, x^5 \, - 10089\, x^6 \, + 39237\, x^7 \, - 13680\, x^8 \, - 1350\, x^9. \qquad \text{(I.38)}$$

These last two order-two operators (I.33) and (I.37) are not linear differential operators with rational coefficients, but with *algebraic* coefficients: there are $\beta(x)^{1/2}$ terms.

One can verify directly that the order-four operator L_4 *can actually be seen as the direct sum of these last two order-two operators* with algebraic coefficients (I.33) and (I.37):

$$L_4 \,\, = \,\, \Bigl(A(x)^{1/2} \cdot \hat{\omega}_{11} \cdot \frac{1}{A(x)^{1/2}}\Bigr) \,\oplus\, \Bigl(\frac{1}{A(x)^{1/2}} \cdot \hat{\omega}_{11} \cdot A(x)^{1/2}\Bigr)$$
$$= \,\, (\Omega_{11} \, - \frac{\Omega_1}{2}) \,\oplus\, (\Omega_{11} \, + \frac{\Omega_1}{2}). \qquad \text{(I.39)}$$

Note that this result is a particular case of a more general result. Let us consider the direct-sum of the two order-two operators with algebraic coefficients depending on one parameter u

$$L_4(u) \,\, = \,\, (\Omega_{11} \, + u \cdot \frac{\Omega_1}{2}) \,\oplus\, (\Omega_{11} \, - u \cdot \frac{\Omega_1}{2}). \qquad \text{(I.40)}$$

The order-four operator is actually a linear differential operator with rational coefficients for any u. It is of the form

$$L_4(u) \,\, = \,\, M_2 \cdot \Omega_{11} \, + u^2 \cdot M_1 \cdot N_1, \qquad \text{(I.41)}$$

where M_2 and M_1 are respectively order-two and order-one operators with rational coefficients and the order-one operator N_1 reads:

$$N_1 \,\, = \,\, D_x \, - \, \frac{1}{8} \cdot \frac{d\ln(\rho(x))}{dx}, \qquad \text{with:} \qquad \rho(x) \,\, = \,\, \frac{\delta(x)^5}{x^4 \cdot \zeta(x)}. \qquad \text{(I.42)}$$

We have the following general result: the exterior square of a direct sum of the form (I.39) is a direct sum of three order-one operators and the symmetric square of an order-two linear differential operator. Let us denote W^+, W^- and W the Wronskian of respectively

$$\hat{\omega}_{11}^{+} \,\, = \,\, A(x)^{1/2} \cdot \hat{\omega}_{11} \cdot \frac{1}{A(x)^{1/2}}, \qquad \hat{\omega}_{11}^{-} \,\, = \,\, \frac{1}{A(x)^{1/2}} \cdot \hat{\omega}_{11} \cdot A(x)^{1/2}, \qquad \hat{\omega}_{11}.$$

One has the following direct sum decomposition for an arbitrary order-two linear differential operator $\hat{\omega}_{11}$:

$$ext^2\Bigl(\hat{\omega}_{11}^{+} \,\oplus\, \hat{\omega}_{11}^{-}\Bigr) \,\, = \qquad \text{(I.43)}$$
$$\Bigl(D_x \, - \frac{d\ln(W^+)}{dx}\Bigr) \oplus \Bigl(D_x \, - \frac{d\ln(W^-)}{dx}\Bigr) \oplus \Bigl(D_x \, - \frac{d\ln(W)}{dx}\Bigr) \oplus M_2^{(3)},$$

where the order-three operator $M_2^{(3)}$ can, for instance, be written as a conjugation of a symmetric square (see also (I.30)):

$$M_2^{(3)} \,\, = \,\, Sym^2\Bigl(\frac{dA(x)}{dx}^{1/2} \, \hat{\omega}_{11}^{-} \cdot \Bigl(\frac{dA(x)}{dx}\Bigr)^{-1/2}\Bigr). \qquad \text{(I.44)}$$

The order-one operator $D_x - \frac{d\ln(W)}{dx}$ in (I.43) actually corresponds to (I.33) together with (I.24). However we see from (I.43) that the irreducible order-two operator $M_2^{(1)}$, we found in the direct sum decomposition (I.23), can in fact be decomposed in a direct sum of order-one operators with *algebraic* coefficients.

Appendix I.1.6. Back to Golyshev and Stienstra operator

Do note that the order-two operator $\mathcal{H}_{11}$, previously encountered with a genus-one situation (see (I.1)), is conjugated to a pullback of (I.29):

$$\mathcal{H}_{11} \quad = \quad \frac{1}{\sqrt{5+3x}} \cdot \, \tilde{\omega}_{11}\Big(x \, \rightarrow \, \frac{5\,x}{5\,+\,3\,x}\Big) \cdot \, \sqrt{5+3x}. \tag{I.45}$$

The pullback of $\mathcal{H}_{11}$ by $\, x \, \rightarrow \, 5\,x \,$ reads:

$$\mathcal{H}_{11}\Big(x \, \rightarrow \, 5\,x\Big) \quad = \quad D_x^2 \quad + \frac{4512\,x^4 + 630\,x^3 + 1232\,x^2 + 102\,x - 1}{(188\,x^3 + 8\,x^2 + 76\,x - 1) \cdot \, (8\,x + 1) \cdot \, x} \cdot \, D_x \\ + \, \frac{6\,(188\,x^3 + 21\,x^2 + 28\,x + 1)}{(188\,x^3 + 8\,x^2 + 76\,x - 1) \cdot \, (8\,x + 1) \cdot \, x}, \tag{I.46}$$

which has as a solution the series with *integer* coefficients:

$$1 \; + 6\,x \; + 204\,x^2 \, + 8790\,x^3 \; + 445170\,x^4 \, + 24577236\,x^5 \; + 1436107596\,x^6 \\ + 87310665684\,x^7 \, + 5466252149820\,x^8 \quad + \, \cdots \tag{I.47}$$

Appendix I.1.7. Hadamard products

The Hadamard square of ω_{11} is a linear differential operator of order ten. The Hadamard square of $\mathcal{H}_{11}$ (or its pullback by $\, x \, \rightarrow \, 5\,x$) is also a linear differential operator of order ten. This order-ten operator is not MUM.

Its head polynomial is

$$x^6 \cdot \, (1 \, - 64\,x) \cdot \, (96256\,x^3 - 512\,x^2 + 608\,x + 1) \cdot \, (35344\,x^3 + 28512\,x^2 + 5792\,x - 1) \\ \times \, (35344\,x^3 - 14288\,x^2 - 8\,x - 1) \cdot \, P_{29},$$

where $\, P_{29} \,$ is a polynomial of degree 29. One verifies easily that Hadamard's theorem [62] on the location of the singularities is verified. If one denotes $\, x_1, \, x_2$, x_3, the three roots of polynomial $\, 188\,x^3 + 8\,x^2 + 76\,x - 1$ (see (I.46)), one sees that the three roots of polynomial $\, 96256\,x^3 - 512\,x^2 + 608\,x + 1$ are nothing but $\, -x_1/8, \, -x_2/8$, $-x_3/8$, the three roots of polynomial $\, 35344\,x^3 + 28512\,x^2 + 5792\,x - 1$ are nothing but $x_1^2, \, x_2^2, \, x_3^2$, the three roots of polynomial $\, 35344\,x^3 - 14288\,x^2 - 8\,x - 1$ are nothing but $\, x_1\,x_2, \, x_2\,x_3, \, x_1\,x_3$, and of course, besides $\, x \, = \, 0 \,$ and $\, x \, = \, \infty, \, 1/64 \, = \, (-1/8)^2$.

Appendix I.2. Other higher genus modular forms

Similar calculations can be performed for the modular forms associated with genus-one modular curves, corresponding, for instance to $\, \tau \, \rightarrow \, N\,\tau$, for $\, N \, = \, 17, 19$. For $N \, = \, 23$ the modular curve is a genus-two curve [150, 151]. These detailed analysis and calculations will be given in a forthcoming publication.

The $\, N \, = \, 23 \,$ genus-two case requires to introduce the order-two operator ‡:

$$\tilde{\omega}_{23}(x) \quad = \quad D_x^2 \quad + \, \frac{1 \, - 12\,x \, + 4\,x^2 \, + 5\,x^3 \, - 33\,x^4 \, + 35\,x^5 \, - 28\,x^6}{x \cdot \, (1 \, - 8\,x \, + 3\,x^2 \, - 7\,x^3) \cdot \, (1 \, - x^2 \, + x^3)} \cdot \, D_x \\ - \, \frac{1 \, - \, x \, - \, x^2 \, + 12\,x^3 \, - 15\,x^4 \, + 14\,x^5}{x \cdot \, (1 \, - 8\,x \, + 3\,x^2 \, - 7\,x^3) \cdot \, (1 \, - x^2 \, + x^3)},$$

This operator has the analytic solution with integer coefficients:

$$1 \; + x \; + 3\,x^2 \; + 13\,x^3 \; + 67\,x^4 \; + 375\,x^5 \; + 2223\,x^6 \; + 13713\,x^7 \; + 87123\,x^8 \; + \; \cdots$$

‡ where one notes that the polynomial $\, 1 - x^2 + x^3 \,$ has the root $\, -1/P$, where $\, P \, = \, 1.324717958 \ldots$ is the smallest *Pisot number* [152].

The pullback of $\tilde{\omega}_{23}(x)$ by $x \rightarrow 1/x$ gives an order-two operator with two analytic solutions (no logarithm):

$$x \quad + \frac{5}{14}\,x^2 \quad + \frac{11}{196}\,x^3 \; - \frac{85}{1372}\,x^4 \; - \frac{3499}{57624}\,x^5 \; - \frac{2041}{57624}\,x^6 \; - \frac{18317}{672280}\,x^7 \\ - \frac{332455}{19765032}\,x^8 \; + \frac{21994361}{1383552240}\,x^9 \quad + \; \cdots, \qquad (I.48)$$

and

$$x^2 \; + \frac{5}{14}\,x^3 \; - \frac{3}{98}\,x^4 \; - \frac{251}{1372}x^5 \; - \frac{137}{1372}\,x^6 \; - \frac{507}{9604}\,x^7 \; - \frac{24007}{470596}\,x^8 \\ - \frac{144083}{6588344}\,x^9 \quad + \; \cdots \qquad (I.49)$$

These two series are *not globally bounded* (but a linear combination of these two series *may be* globally bounded ...). The fact that the two previous solutions have no logarithmic terms *excludes any relation like* (149). We encounter the same situation with the order-two operators $\tilde{\omega}_{11}(x)$, $\tilde{\omega}_{17}(x)$, $\tilde{\omega}_{19}(x)$, $\tilde{\omega}_{29}(x)$, $\tilde{\omega}_{31}(x)$, $\tilde{\omega}_{41}(x)$, $\tilde{\omega}_{47}(x)$, $\tilde{\omega}_{59}(x)$, and $\tilde{\omega}_{71}(x)$, the corresponding two series at $x = \infty$ having no logarithmic terms yielding the same obstruction for a relation like (149). These various order-two operators correspond to higher order genus modular curves [153], namely *genus-one* for $\tilde{\omega}_{11}(x)$, $\tilde{\omega}_{17}(x)$, $\tilde{\omega}_{19}(x)$, *genus-two* for $\tilde{\omega}_{29}(x)$, $\tilde{\omega}_{31}(x)$, *genus-three* for $\tilde{\omega}_{41}(x)$, *genus-four* for $\tilde{\omega}_{47}(x)$, *genus-five* for $\tilde{\omega}_{59}(x)$, and *genus-six* for $\tilde{\omega}_{71}(x)$.

Note that all these higher-genus $\tilde{\omega}_n(x)$'s are simply homomorphic to their adjoint. They are such that

$$R(x)^{1/M} \cdot adjoint(\tilde{\omega}_n(x)) \quad = \quad \tilde{\omega}_n(x) \cdot R(x)^{1/M}, \qquad (I.50)$$

where $R(x)$ is a rational function, and where $M = 2$, except for $n = 19$ where $M = 6$.

Appendix I.3. Order-two operators associated with modular Atkin equations

Let us give the explicit expressions of some ω_n's for $n = 17, 19, 23, 31, 39, 41, 47, 59, 71$. The higher genus ω_n's read

$$\omega_i \quad = \quad D_x^2 \; + \frac{A_i}{C_i} D_x \quad + \frac{B_i}{C_i},$$

with:

$$A_{11} \; = \; 84\,x^4 - 120\,x^3 - 16\,x^2 + 96\,x - 4,$$
$$B_{11} \; = \; 3\,(x-1)\,(7\,x^2 - x - 2),$$
$$C_{11} \; = \; 4\,x \cdot\,(7\,x^3 - 19\,x^2 + 17\,x - 1)\,(x+1),$$

$$A_{17} \; = \; 448\,x^5 - 576\,x^4 + 280\,x^3 + 224\,x^2 - 264\,x + 16,$$
$$B_{17} \; = \; 168\,x^4 - 180\,x^3 + 67\,x^2 + 41\,x - 20,$$
$$C_{17} \; = \; 16\,x(x-1)(8\,x^4 - 4\,x^3 + 3\,x^2 + 10\,x - 1),$$

$$A_{19} \; = \; 264\,x^5 - 354\,x^4 - 258\,x^3 + 300\,x^2 + 93\,x - 9,$$
$$B_{19} \; = \; 110\,x^4 - 130\,x^3 - 69\,x^2 + 56\,x + 6,$$
$$C_{19} \; = \; 9\,x \cdot\,(2\,x+1)\,(x+1)\,(4\,x^3 - 12\,x^2 + 10\,x - 1),$$

$$A_{23} \;=\; 28\,x^6-35\,x^5+33\,x^4-5\,x^3-4\,x^2+12\,x-1,$$
$$B_{23} \;=\; 14\,x^5-15\,x^4+12\,x^3-x^2-x+1,$$
$$C_{23} \;=\; x\cdot\,(7\,x^3-3\,x^2+8\,x-1)(x^3-x^2+1),$$

$$A_{29} \;=\; 504\,x^7-64\,x^6-896\,x^5-480\,x^4+400\,x^3+512\,x^2+72\,x-16,$$
$$B_{29} \;=\; 315\,x^6-35\,x^5-441\,x^4-213\,x^3+109\,x^2+103\,x+4,$$
$$C_{29} \;=\; 16\,x\cdot\,(x+1)(7\,x^6-8\,x^5-8\,x^4-2\,x^3+12\,x^2+4\,x-1),$$

$$A_{31} \;=\; 108\,x^6+343\,x^5+477\,x^4+235\,x^3+28\,x^2-6\,x-1,$$
$$B_{31} \;=\; 60\,x^5+161\,x^4+180\,x^3+69\,x^2+5\,x-1,$$
$$C_{31} \;=\; x\cdot\,(x^3+3\,x^2+4\,x+1)(27\,x^3+17\,x^2-1)$$

$$A_{41} \;=\; 704\,x^9+960\,x^8-360\,x^7-1472\,x^6-672\,x^5+480\,x^4$$
$$+\,720\,x^3-128\,x^2-120\,x+16,$$
$$B_{41} \;=\; 616\,x^8+756\,x^7-289\,x^6-969\,x^5-353\,x^4+265\,x^3$$
$$+\,261\,x^2-39\,x-12,$$
$$C_{41} \;=\; 16\,x\cdot\,(x-1)(8\,x^8+20\,x^7+15\,x^6-8\,x^5-20\,x^4-10\,x^3$$
$$+\,8\,x^2+4\,x-1),$$

$$A_{47} \;=\; 66\,x^{10}-154\,x^9+190\,x^8-135\,x^7+52\,x^6+56\,x^5-57\,x^4$$
$$+\,60\,x^3-22\,x^2+9\,x-1,$$
$$B_{47} \;=\; 66\,x^9-140\,x^8+158\,x^7-103\,x^6+35\,x^5+33\,x^4$$
$$-\,28\,x^3+24\,x^2-6\,x+1,$$
$$C_{47} \;=\; x\cdot\,(11\,x^5-6\,x^4+15\,x^3-5\,x^2+5\,x-1)\,(x^5-2\,x^4+x^3$$
$$+\,x^2-x+1),$$

$$A_{59} \;=\; 308\,x^{12}+624\,x^{11}+1632\,x^{10}+2640\,x^9+3520\,x^8+3816\,x^7$$
$$+\,3232\,x^6+2128\,x^5+1008\,x^4+280\,x^3-24\,x-4,$$
$$B_{59} \;=\; 385\,x^{11}+720\,x^{10}+1748\,x^9+2600\,x^8+3128\,x^7+3024\,x^6$$
$$+\,2217\,x^5+1216\,x^4+448\,x^3+86\,x^2-4\,x-4,$$
$$C_{59} \;=\; 4\,x\cdot\,(x^3+2\,x+1)\,(11\,x^9+24\,x^8+46\,x^7+61\,x^6+60\,x^5$$
$$+\,44\,x^4+21\,x^3+4\,x^2-2\,x-1),$$

$$A_{71} \;=\; 88\,x^{14}-30\,x^{13}-280\,x^{12}-195\,x^{11}+420\,x^{10}+671\,x^9-5\,x^8$$
$$-\,666\,x^7-444\,x^6+91\,x^5+231\,x^4+95\,x^3+4\,x^2-6\,x-1,$$
$$B_{71} \;=\; 132\,x^{13}-42\,x^{12}-372\,x^{11}-248\,x^{10}+475\,x^9+717\,x^8$$
$$+\,25\,x^7-563\,x^6-362\,x^5+29\,x^4+112\,x^3+38\,x^2+x-1,$$
$$C_{71} \;=\; x\cdot\,(11\,x^7-4\,x^6-18\,x^5-5\,x^4+11\,x^3+7\,x^2-1)\,(x^7-2\,x^5$$
$$-\,3\,x^4+x^3+5\,x^2+4\,x+1).$$

Appendix J. Yukawa coupling as ratio of determinants

Consider an order-four MUM linear differential operator. Let us introduce the determinantal variables $W_m = \det(M_m)$ which are the determinants‡ of the following $m \times m$ matrices M_m, $m = 1, \cdots, 4$, with entries expressed in terms of derivatives of the four solutions $y_0(x)$, $y_1(x)$, $y_2(x)$ and $y_3(x)$ of the MUM linear differential operator (see Section (5) for the definitions). One takes $W_1(x) = y_0(x)$ and:

$$M_2 = \begin{bmatrix} y_0 & y_1 \\ y_0' & y_1' \end{bmatrix}, \quad M_3 = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_0' & y_1' & y_2' \\ y_0'' & y_1'' & y_2'' \end{bmatrix}, \quad M_4 = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0' & y_1' & y_2' & y_3' \\ y_0'' & y_1'' & y_2'' & y_3'' \\ y_0''' & y_1''' & y_2''' & y_3''' \end{bmatrix},$$

$$\text{where:} \qquad y_i' = \frac{d}{dx} y_i, \qquad y_i'' = \frac{d^2}{dx^2} y_i, \qquad y_i''' = \frac{d^3}{dx^3} y_i. \tag{J.1}$$

Since $q = exp(y_1/y_0)$, and hence,

$$q \cdot \frac{d}{dq} = \frac{W_1^2}{W_2} \cdot \frac{d}{dx} = \frac{y_0^2}{W_2} \cdot \frac{d}{dx}, \tag{J.2}$$

and thus

$$\Bigl(q \cdot \frac{d}{dq}\Bigr)^2 = \frac{y_0^4}{W_2^2} \cdot \frac{d^2}{dx^2} + 2\, \frac{y_0^3}{W_2^2} \frac{dy_0}{dx} \cdot \frac{d}{dx} - \frac{y_0^4}{W_2^3} \frac{dW_2}{dx} \cdot \frac{d}{dx}, \tag{J.3}$$

we deduce, after some simple algebra, an alternative definition for the *Yukawa coupling*:

$$K(q) = \Bigl(q \cdot \frac{d}{dq}\Bigr)^2 \Bigl(\frac{y_2}{y_0}\Bigr) = \frac{W_1^3 \cdot W_3}{W_2^3} = \frac{y_0^3 \cdot W_3}{W_2^3}. \tag{J.4}$$

to be compared with the other previous alternative expression previously given (170) for the *Yukawa coupling*

$$K(q) = \frac{x(q)^3 \cdot W_4^{1/2}}{y_0^2} \cdot \Bigl(\frac{q}{x(q)} \cdot \frac{dx(q)}{dq}\Bigr)^3 = \frac{W_4^{1/2}}{y_0^2} \cdot \Bigl(q \cdot \frac{dx(q)}{dq}\Bigr)^3. \tag{J.5}$$

In fact from (J.2) we deduce

$$\Bigl(q \cdot \frac{dx(q)}{dq}\Bigr)^3 = \frac{W_1^6}{W_2^3} = \frac{y_0^6}{W_2^3}, \tag{J.6}$$

and, so, (J.5) is compatible with (J.4) *if the following identity is verified*:

$$W_3^2 = W_4 \cdot y_0^2 = W_4 \cdot W_1^2. \tag{J.7}$$

This identity is in fact specific of *order-four operators conjugated to their adjoint* (see below (J.17)). Therefore we prefer to use definition (J.4) for the Yukawa coupling, instead of the more restricted definition (J.5).

Let us assume that the pullback $p(x)$ has a series expansion of the form

$$p(x) = \lambda \cdot x^r \cdot A(x), \tag{J.8}$$

where the exponent r is an integer, where λ is a constant, and where $A(x)$ is a function analytic at $x = 0$ with the series expansion:

$$A(x) = 1 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \cdots$$

‡ For an order-four operator the Wronskian is W_4.

The determinantal variables W_m's transform very nicely under pullbacks $p(x)$ of the form (J.8):

$$(W_1(x),\, W_2(x),\, W_3(x),\, W_4(x)) \quad \longrightarrow \tag{J.9}$$
$$\Big(W_1(p(x)),\ \frac{p'}{r} \cdot W_2(p(x)),\ \frac{p'^3}{r^3} \cdot W_3(p(x)),\ \frac{p'^6}{r^6} \cdot W_4(p(x))\Big), \qquad p' \, = \, \frac{dp(x)}{dx}.$$

One can show that the nome (51) of an order- N operator transforms under a pullback $p(x)$:

$$q(x) \,\longrightarrow\, Q(x) \qquad \text{with:} \qquad \lambda \cdot Q(x)^r \,=\, q(p(x)). \tag{J.10}$$

From the covariance property (J.9), and from the previous transformation $q \rightarrow \lambda \cdot q^r$ for the nome, one easily gets the transformation of the Yukawa coupling seen as a function of the nome $K(q) \,\rightarrow\, K(\lambda \cdot q^r)$:

$$K(q(x)) \,=\, \frac{W_1(x)^3 \cdot W_3(x)}{W_2(x)^3} \tag{J.11}$$
$$\longrightarrow \quad \frac{W_1(p(x))^3 \cdot W_3(p(x))}{W_2(p(x))^3} \,=\, K(q(p(x))) \,=\, K(\lambda \cdot Q(x)^r).$$

For $\lambda \,=\, 1$ and $r \,=\, 1$ (i.e. when the pullback is a deformation of the identity transformation), one recovers the known invariance of the Yukawa coupling by pullbacks (see Proposition 3 in [154]).

One finds *another* pullback invariant ratio, namely:

$$K^\star \,=\, \frac{W_1 \cdot W_3^3}{W_4 \cdot W_2^3}, \tag{J.12}$$

which is, in fact, *nothing but the Yukawa coupling for the adjoint* of the original operator.

Another invariance property is worth noting. Let us consider two linear differential operators Ω_1 and Ω_2 of order N that are equivalent, in the sense of the equivalence of linear differential operators. This means that there exists linear differential operators intertwiners $I_1,\ I_2,\ J_1,\ J_2$, of order at most $N-1$ such that

$$\Omega_1 \cdot I_1 \,=\, I_2 \cdot \Omega_2, \qquad \text{and:} \qquad J_1 \cdot \Omega_1 \,=\, \Omega_2 \cdot J_2. \tag{J.13}$$

Let us assume that one of these intertwiners is a linear differential operator of order zero (a function), then the previous homomorphism between operators amounts to saying that the two operators are conjugated by a function:

$$\Omega_2 \,=\, \rho(x) \cdot \Omega_1 \cdot \rho(x)^{-1}, \tag{J.14}$$

which correspond to changing the four solutions as follows: $y_i \,\rightarrow\, \rho(x) \cdot y_i$. In such a case (quite frequent as will be seen in forthcoming publication) the previous determinant variables transform, again, very nicely under the "gauge" function $\rho(x)$:

$$(W_1,\, W_2,\, W_3,\, W_4) \quad \rightarrow \quad (\rho(x) \cdot W_1,\ \rho(x)^2 \cdot W_2,\ \rho(x)^3 \cdot W_3,\ \rho(x)^4 \cdot W_4). \tag{J.15}$$

It is straightforward to see that the Yukawa coupling and the "dual Yukawa", $K^\star$, are *invariant by such a transformation*¶. Two conjugated operators (J.14) automatically have the same Yukawa coupling.

Do note that the Yukawa couplings for two operators, which are non trivially homomorphic to each other (intertwiners of order one, two, ...), are actually different.

¶ K and $K^\star$ (and their combinations) are the only monomials $W_1^{n_1}\, W_2^{n_2}\, W_3^{n_3}\, W_4^{n_4}$ to be invariant by (J.9) and (J.15).

The (pullback invariant) Yukawa coupling is not preserved by operator equivalence (see subsection (8.9.3)).

Remark: The definition of these determinantal variables W_i's heavily relies on the MUM structure of the operator†. The four solutions are not on the same footing: the log filtration *imposes a natural order* between the four solutions the definition of W_i's relies on. It is worth noting that if one permutes the four solution y_i, one would get 24 other sets of (W_1, W_2, W_3, W_4) which are actually *also nicely covariant by pullbacks*, thus yielding a finite set of other "Yukawa couplings" or adjoint Yukawa coupling $K^\star$ *also invariant by pullbacks*. In fact these "Yukawa couplings", and other $K^\star$, can even be defined when the linear differential operator is not MUM, and they are still invariant by pullbacks.

Appendix J.1. Pullback-invariants for higher order ODEs

These simple calculations can straightforwardly be generalised to higher order linear differential equations. We give here the invariants for higher order linear differential operators.

Let us give, for the n-th order linear differential operator the list of the K_n invariants by pullback transformations:

$$K_3 \,=\, \frac{W_1^3 \cdot W_3}{W_2^3}, \qquad K_4 \,=\, \frac{W_1^8 \cdot W_4}{W_2^6}, \qquad K_5 \,=\, \frac{W_1^{15} \cdot W_5}{W_2^{10}}, \qquad K_6 \,=\, \frac{W_1^{24} \cdot W_6}{W_2^{15}}, \quad \cdots$$

$$K_n \,=\, \frac{W_1^{a_n} \cdot W_n}{W_2^{b_n}}, \qquad \text{with:} \qquad a_n \,=\, n \cdot (n-2), \qquad b_n \,=\, \frac{n \cdot (n-1)}{2}. \qquad (J.16)$$

A n-th order linear differential operator has K_n as an invariant by pullback transformation, as well as all the K_m with $m \le n$. K_3 is the Yukawa coupling, and one remarks, for the order-four operators, that the other pullback invariant $K^\star$ (see (J.12)), which is actually also the Yukawa coupling of the adjoint operator, is nothing but K_3^3/K_4.

For order-four operators conjugated to their adjoint (see (J.14)) (i.e. operators homomorphic to their adjoint, the intertwiner being an order zero differential operator, a function), one has the equality

$$K_4 \,=\, K_3^2, \qquad \text{i.e.} \qquad K_3 \,=\, K^\star, \qquad \text{or} \qquad W_3^2 \,=\, W_1^2 \cdot W_4, \qquad (J.17)$$

to be compared with the equality in Almkvist et al. (see Proposition 2 in [88])

$$y_0\, y_3' \,-\, y_3\, y_0' \quad = \quad y_1\, y_2' \,-\, y_2\, y_1', \qquad (J.18)$$

which is satisfied when the Calabi-Yau condition that the exterior square is of order five is satisfied.

If a linear differential operator Ω_4 verifies condition (J.17), its conjugate by a function, $\rho(x) \cdot \Omega_4 \cdot \rho(x)^{-1}$, also verifies condition (J.17) (their Yukawa couplings are equal).

The condition (J.17) is not satisfied for linear differential operators homomorphic to their adjoint with *non-trivial* intertwiner (of order greater than zero). For instance the order-four operator (171) does not satisfy condition (J.17).

† Or recalling the powerful result of Steenbrink [143] that the filtration by the logs is the same as the Hodge filtration, the definition relies on this filtration.

Remark: Concerning order-three operators and hypergeometric functions. It is worth noting the four examples

$$ {}_3F_2\Big([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}], [1, 1], 64\,x\Big), \qquad {}_3F_2\Big([\frac{1}{2}, \frac{1}{3}, \frac{2}{3}], [1, 1], 108\,x\Big), $$
$$ {}_3F_2\Big([\frac{1}{2}, \frac{1}{4}, \frac{3}{4}], [1, 1], 256\,x\Big), \qquad {}_3F_2\Big([\frac{1}{2}, \frac{1}{6}, \frac{5}{6}], [1, 1], 1728\,x\Big), $$

which are such that their series expansions, as well as their associated nome, mirror map (compositional inverse of the nome), are series with *integer* coefficients, the previous invariant K_3 being *the constant* 1.

This however comes as no surprise since the four corresponding operators are all symmetric squares† of order-two operators. Their solutions of the form $y_0(x) = u(x)^2$, $y_1(x) = u(x) \cdot v(x)$, and $y_2(x) = v(x)^2/2$, give automatically $K_3 = 1$.

Appendix K. Quasi-Calabi-Yau ODE associated to the Hadamard product of two HeunG functions

The operator having the Hadamard product of the two HeunG functions $HeunG(a, q, 1, 1, 1, 1; x)$ and $HeunG(A, Q, 1, 1, 1, 1; x)$ as a solution reads:

$$ (x-1)\,(x-a)\,(x-A)\,(x-A\,a)\,(A\,a-x^2)^2 \cdot x^3 \cdot D_x^4 \\ + 2\,(x^2 - A\,a) \cdot U_3 \cdot x^2 \cdot D_x^3 \quad - U_2 \cdot x \cdot D_x^2 \quad - U_1 \cdot D_x \quad + U_0, \tag{K.1} $$

where:

$$ \begin{aligned} U_3 \;=\;& 5\,x^6 - 4\,(a+1)\,(A+1) \cdot x^5 + \big(3\,aA^2 + 3\,a^2A - Aa + 3\,A + 3\,a\big) \cdot x^4 \\ & + 4\,Aa\,(a+1)\,(A+1) \cdot x^3 - Aa\,\big(5\,a^2A + 5\,aA^2 + 9\,Aa + 5\,a + 5\,A\big) \cdot x^2 \\ & + 4\,A^2a^2\,(a+1)\,(A+1) \cdot x \quad - 3\,a^3A^3, \end{aligned} $$

$$ \begin{aligned} U_2 \;=\;& -25\,x^8 + (14\,Aa + (14+Q)\,a + (14+q)\,A + 14 + q + Q) \cdot x^7 \\ & - 2\,(3\,Aa\,(a+A) + Aa\,(Q+q-29) + 3\,A + qA + 3\,a + Qa) \cdot x^6 \\ & - Aa\,(42\,Aa + 42\,A + qA + Qa + 42\,a + Q + q + 42) \cdot x^5 \\ & + 2\,Aa\,(11\,Aa\,(a+A) + 2\,Aa\,(q+Q-2) + 11\,a + 11\,A + 2\,Qa + 2\,qA) \cdot x^4 \\ & + A^2\,a^2\,(30\,Aa + 30\,a + 30\,A - Qa - qA - Q - q + 30) \cdot x^3 \\ & - 2\,A^2a^2\,(12\,A\,a\,(a+A) + A\,a\,(Q+q+25) + Qa + qA + 12\,a + 12\,A) \cdot x^2 \\ & + A^3a^3\,(14\,A\,a + 14\,a + 14\,A + Qa + qA + Q + q + 14) \cdot x \quad - 7\,A^4a^4, \end{aligned} $$

$$ \begin{aligned} U_1 \;=\;& -15\,x^8 + 2\,(2\,Aa + 2\,A + 2\,a + Qa + qA + Q + q + 2) \cdot x^7 \\ & - 2\,(Aa\,(Q+q-23) + qA + Qa) \cdot x^6 \\ & - 6\,A\,a\,(2\,Aa + 2\,A + 2\,a + Qa + qA + Q + q + 2) \cdot x^5 \\ & - 4\,A\,a\,(A\,a\,(a+A) - 2\,A\,a\,(Q+q-8) - 2\,Qa - 2\,qA + a + A) \cdot x^4 \\ & + 2\,A^2a^2\,(18\,A\,a + Qa + qA + 18\,a + 18\,A + Q + q + 18) \cdot x^3 \\ & - 6\,A^2a^2\,(2\,A\,a\,(a+A) + A\,a\,(Q+q+5) + 2\,A + 2\,a + Qa + qA) \cdot x^2 \\ & + 2\,A^3a^3\,(2\,Aa + 2\,A + 2\,a + Qa + qA + Q + q + 2) \cdot x \quad - A^4a^4, \end{aligned} $$

† Recall that ${}_3F_2([1/2,\, \alpha,\, 1-\alpha], [1, 1], x) = {}_2F_1([\alpha/2,\, 1/2-\alpha/2], [1], x)^2$.

$$U_0 \,=\, x^7 \,-q\,Q\,x^6 \,-\, (-q^2A \,+3\,Aa \,-aQ^2)\cdot\, x^5 \\ -\,a\,A\,(2\,qA\,+2\,q\,+2\,Qa\,+2\,Q\,-qQ)\cdot\,x^4 \\ +\,A\,a\,\big(3\,A\,a\,(2\,Q+2\,q+1)\,+6\,Qa\,+6\,qA-2\,aQ^2-2\,q^2A\big)\cdot\,x^3 \\ -\,A^2a^2\,(6\,qA\,+6\,q\,+6\,Qa\,+6\,Q\,-qQ)\cdot\,x^2 \\ +\,A^2a^2\,\big(A\,a\,(2\,Q+2\,q-1)\,+2\,Qa+2\,qA\,+aQ^2\,+q^2A\big)\cdot\,x\,\,-q\,Q\,a^3A^3.$$

This order-four operator satisfying the Calabi-Yau condition (its exterior square is of order five). The solution of this order-four operator, analytic at $x\,=\,0$ reads:

$$y_0(x) \,=\, 1\,+\frac{Q\,q}{Aa}\cdot\,x\,+\,\frac{\big(2\,qa+2\,q+q^2-a\big)\,\big(2\,QA+2\,Q+Q^2-A\big)}{16\,a^2A^2}\cdot x^2\,+\,\cdots,$$

the nome reads

$$x\quad+\frac{(Qa\,+qA\,+Q\,+q\,-4\,qQ)}{Aa}\cdot\,x^2\quad+\cdots, \tag{K.2}$$

and the first terms of its Yukawa coupling $K(x)$ read:

$$K(x)\,=\,1\,+\frac{(1\,-3\,Q-3\,q\,+a+A\,+10\,qQ+Aa-3\,qA-3\,Qa)}{Aa}\cdot\,x\,+\,\cdots \tag{K.3}$$

Even inside this restricted set of HeunG functions solutions of the form $HeunG(a,\,q,\,1,\,1,\,1,\,1;\,x)$ it is hard to find exhaustively the values of the two parameters a, and of the accessory parameter q, such that the series $HeunG(a,\,q,\,1,\,1,\,1,\,1;\,x)$ is globally bounded. The HeunG function $HeunG(a,\,q,\,1,\,1,\,1,\,1;\,b\,x)$ becomes a series $1\,+N_1\,x\,+N_2\,x^2\,+\,\cdots$ with integer coefficients $N_1,\,N_2$, ..., for

$$a\,=\,q\cdot\,\frac{(4\,N_2\,-N_1^2)\cdot\,q\,-2\,N_1^2}{N_1^2\cdot\,(2\,q\,-1)},\qquad\qquad b\,=\,N_1\cdot\,\frac{a}{q}. \tag{K.4}$$

These are necessary conditions. One can find some other necessary conditions on the integer coefficients of the series. Besides $q\,=\,1/2,\ a\,=\,2\,q,\ a\,=\,1/3\,q\,(13\,q\,-6)/(2\,q\,-1)$ (i.e. $N_2/N_1^2\,=\,5/4$ or $N_2/N_1^2\,=\,4/3$) that require some specific analysis, one finds that one has necessarily the following relation among the first four coefficients $N_1,\,N_2,\,N_3,\,N_4$, namely:

$$27\cdot\,N_3^2\quad+10\,\cdot\,(N_1^2-9\,N_2)\cdot\,N_1\cdot\,N_3\quad+8\cdot\,(4\,N_1^2-3\,N_2)\cdot\,N_4 \\ -\,9\cdot\,N_2^2\cdot\,(N_1^2-6\,N_2)\quad=\quad 0. \tag{K.5}$$

One verifies easily that (K.5) is actually verified for (52) and (100).

Note that the ratio N_p/N_1^p are rational expressions of a and q, invariant by $(a,\,q)\,\rightarrow\,(1/a,\,q/a)$, of the form $P(a,\,q)/q^p$, where $P(a,\,q)$ is a polynomial.

However, finding the values of the accessory parameter q, such that $HeunG(a,\,q,\,1,\,1,\,1,\,1;\,x)$ is globally bounded remains difficult. Is it possible to finds such values of the accessory parameter q for any given integers N_1 and N_2 ? For instance for $N_1\,=\,2,\ N_2\,=\,10$ the series with integer coefficients $HeunG(-8,\,q,\,1,\,1,\,1,\,1;\,8\,x)$ is actually a series with integer coefficients for $q\,=\,-2$.

Furthermore it is difficult to find the values of the two parameters a and q such that the order-two operator having $HeunG(a,\,q,\,1,\,1,\,1,\,1;\,x)$ as a solution is globally nilpotent. For instance, if one restricts to $a\,=\,9/8$, it is hard to show that the only rational number value of the accessory parameter q yielding global nilpotence is $q\,=\,3/4$. For q a rational number one gets an infinite set of divisibility conditions. The prime 5 must divide the numerator of $(q+3)\,(q+1)\,(q+2)\,(q^2+4\,q+1)$, the prime

7 must divide the numerator of $(q^2+6\,q+6)\,(q+6)\,(q+1)\,(q+4)\,(q^2+4\,q+1)$, etc. With $q\,=\,N/D$ where N and D integers <1000 and with the conditions emerging from the p-curvature calculations for the first primes ≤ 42. One finds that $q\,=\,3/4$ is the only value of the accessory parameter corresponding to global nilpotence.

Restricting to HeunG functions solutions of the form $HeunG(a,\,q,\,1,\,1,\,1,\,1;\,60\,a\,x)$ for integer values of a and q it is hard to find the integer values of a and q such that the corresponding series are globally bounded.

Appendix L. Yukawa coupling of Calabi-Yau ODEs

Let us give the expansion of the Yukawa coupling for a set of other $H_{m,n}$ that are Calabi-Yau: in particular they are MUM and their exterior squares are of order *five* (the "Calabi-Yau condition").

For $H_{4,8}$ the Yukawa coupling reads:

$$K(q) \;=\; K^\star(q) \;=\; 1 \;+\,16\cdot q \;+352\cdot q^2 \;+33280\cdot q^3 \;+\,2058528\cdot q^4 \\ +\,123766016\cdot q^5 \;+\,7347718144\cdot q^6 \,+\,439489011712\cdot q^7 \\ +\,26579639900960\cdot q^8 \;+\,1616513123552128\cdot q^9 \;+\;\cdots, \tag{L.1}$$

which is Number 36 in Almkvist et al. large tables of Calabi-Yau ODEs [91].

For $H_{4,9}$ the Yukawa coupling reads:

$$K(q) \;=\; K^\star(q) \;=\; 1 \;+\,12\cdot q \;-324\cdot q^2 \;-29544\cdot q^3 \;-1314756\cdot q^4 \\ -\,12971988\cdot q^5 \;+\,2033927928\cdot q^6 \;+\,146587697352\cdot q^7 \\ +\,4172739566652\cdot q^8 \;-77469253445544\cdot q^9 \;+\;\cdots, \tag{L.2}$$

which is Number 133 in tables [91].

For $H_{6,6}$ it reads:

$$K(q) \;=\; K^\star(q) \;=\; 1 \;+\,37\cdot q \;-4523\cdot q^2 \;+412327\cdot q^3 \;-33924139\cdot q^4 \\ +\,2662557912\cdot q^5 -203154013049\cdot q^6 \;+\,15217617773948\cdot q^7 \\ -\,1125153432893483\cdot q^8 \;+82390368380951296\cdot q^9 \;+\;\cdots, \tag{L.3}$$

which is Number 144 in tables [91].

For $H_{6,8}$ it reads:

$$K(q) \;=\; K^\star(q) \;=\; 1 \;+\,24\cdot q \;-2012\cdot q^2 \;+139056\cdot q^3 \;-8227932\cdot q^4 \\ +\,468328024\cdot q^5 \;-25856580632\cdot q^6 \;+\,1402012096656\cdot q^7 \\ -\,74994891745116\cdot q^8 \;+3972880128014736\cdot q^9 \;+\;\cdots, \tag{L.4}$$

which is Number 176 in tables [91].

For $H_{6,9}$ it reads:

$$K(q) \;=\; K^\star(q) \;=\; 1 \;+\,18\cdot q \;-1674\cdot q^2 \;+88209\cdot q^3 \;-4801770\cdot q^4 \\ +\,239279643\cdot q^5 \;-11680323039\cdot q^6 \;+\,558685593414\cdot q^7 \\ -\,26379917556714\cdot q^8 \;+1233104626297710\cdot q^9 \;+\;\cdots, \tag{L.5}$$

which is Number 178 in tables [91].

For $H_{8,8}$:

$$K(q) \;=\; K^\star(q) \;=\; 1 \;+\,16\cdot q \;-864\cdot q^2 \;+47104\cdot q^3 \;-1890528\cdot q^4 \\ +\,80502016\cdot q^5 \;-3118639104\cdot q^6 \;+\,123287486464\cdot q^7 \\ -\,4691784791264\cdot q^8 \;+179585946086272\cdot q^9 \;+\;\cdots, \tag{L.6}$$

which is Number 107 in tables [91].

For $H_{8,9}$ the Yukawa coupling reads:

$$K(q) \,\, = \,\, K^\star(q) \,\, = \,\, 1 \, + 12 \cdot q \, - 756 \cdot q^2 \, + 27192 \cdot q^3 \, - 1144644 \cdot q^4 \\ + 39948012 \cdot q^5 \, - 1377082728 \cdot q^6 \, + 47882164776 \cdot q^7 \\ - 1608623259588 \cdot q^8 \, + 53732432848152 \cdot q^9 \, + \, \cdots, \qquad (L.7)$$

which is Number 163 in tables [91], and for $H_{9,9}$ it reads:

$$K(q) \,\, = \,\, K^\star(q) \,\, = \,\, 1 \, + 9 \cdot q \, - 567 \cdot q^2 \, + 20205 \cdot q^3 \, - 615735 \cdot q^4 \\ + 19431009 \cdot q^5 \, - 608213043 \cdot q^6 \, + 18406651167 \cdot q^7 \\ - 542566460727 \cdot q^8 \, + 15865350996861 \cdot q^9 \, + \, \cdots, \qquad (L.8)$$

which is Number 165 in tables [91].

Appendix M. Modular form character of ${}_2F_1([1/6,\, 1/6],\, [1],\, x)$

The modular form character of (152) is clear on the remarkable (and intriguing) identity (58) (or (59)) in Maier [123]:

$$ {}_2F_1\Big([\frac{1}{6},\, \frac{1}{6}],\, [1],\, \mathcal{P}_1\Big) \qquad (M.1)$$
$$ = \,\, 2 \cdot \Big(\frac{(x+60)\,(x+80)\,(x+96)}{(x+48)\,(x+120)^2}\Big)^{-1/6} \cdot {}_2F_1\Big([\frac{1}{6},\, \frac{1}{6}],\, [1],\, \mathcal{P}_2\Big),$$

where the two pullbacks read respectively ($\circ$ denotes the composition of functions)

$$\mathcal{P}_1 \,\, = \,\, \frac{-1}{432} \cdot \frac{x\,(x+60)^2\,(x+72)^2\,(x+96)}{(x+48)(x+80)(x+120)^2} \qquad (M.2)$$
$$ = \,\, \frac{x}{x-1} \circ \frac{1728\,x}{(x\,+16)^3} \circ G \,\, = \,\, \frac{x}{x-1} \circ \frac{1728\,x^2}{(x\,+256)^3} \circ \frac{64^2}{G},$$
$$\mathcal{P}_2 \,\, = \,\, \frac{-1}{432} \cdot \frac{x^2(x+48)^2(x+72)(x+120)}{(x+60)\,(x+80)^2\,(x+96)^2} \qquad (M.3)$$
$$ = \,\, \frac{x}{x-1} \circ \frac{1728\,x}{(x\,+16)^3} \circ \frac{64^2}{H} \,\, = \,\, \frac{x}{x-1} \circ \frac{1728\,x^2}{(x\,+256)^3} \circ H,$$

where

$$G \,\, = \,\, \frac{8\,x \cdot\, (x+96)}{(x+72)(x+60)}, \qquad H \,\, = \,\, \frac{-64\,x \cdot\, (x+48)}{(x+72)\,(x+120)}, \qquad (M.4)$$

The algebraic relation between these two pullbacks $(u,\, v) \, = \, (\mathcal{P}_1,\, \mathcal{P}_2)$, corresponds to the (genus zero) *modular curve*:

$$6912\, u\, v \cdot\, (u^4\, v^4 + 1) \,\,\, - 3\, u\, v \cdot\, \Big(80\,(u^2 + \, v^2) \, + 529999\, u\, v\Big) \cdot\, (v^2\, u^2 + 1) \\ + 16\,(u+v) \cdot\, (u\, v + 1) \cdot\, (u^2 + v^2 \, - 241\, u\, v) \cdot\, (v^2\, u^2 \, - u\, v \, + 1) \\ - 6\, u\, v \cdot\, (u+v) \cdot\, (u\, v + 1) \cdot\, \Big(640\,(u^2 + v^2) \, - 652959\, u\, v\Big) \\ + u\, v \cdot\, \Big(6912\,(u^4 + v^4) \, - 1589997\,(u^3\, v + v^3\, u) \, + 21958300\, v^2\, u^2\Big) \,\, = \,\, 0.$$

Other identities are worth noting on these pullbacks, for instance:

$$\frac{-4\,x}{(1-x)^2} \circ \mathcal{P}_1 \;=\; \frac{1728x}{(x+16)^3} \circ F \;=\; \frac{1728x^2}{(x+256)^3} \circ \frac{4096}{F}, \qquad \text{and:}$$

$$\frac{-4\,x}{(1-x)^2} \circ \mathcal{P}_2 \;=\; \frac{1728x^2}{(x+256)^3} \circ F \;=\; \frac{1728x}{(x+16)^3} \circ \frac{4096}{F}, \qquad \text{where:}$$

$$F \;=\; \frac{x \cdot (x+80)\,(x+48)\,(x+96)}{(x+120)\,(x+72)\,(x+60)}.$$

It can also be illustrated through the identity:

$$
\begin{aligned}
&{}_2F_1\Big([\frac{1}{6}, \frac{1}{6}], [1], \frac{27\,x}{(1+8\,x)^2\,(1-x)}\Big) \qquad \text{(M.5)}\\
&= \frac{(1-x)^{1/6}\,(1+8\,x)^{1/3}}{(1+7\,x+x^2)^{1/4}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27\,x^2\,(1-x)^2\,(1+8\,x)}{4\,(1+7\,x+x^2)^3}\Big)\\
&= \frac{(1-x)^{1/6}\,(1+8\,x)^{1/3}}{(1-4\,x)^{1/2}} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{-108\,x\,(1-x)\,(1+8\,x)^2}{(1-4\,x)^6}\Big),
\end{aligned}
$$

where the last two pullbacks are related by the fundamental modular curve [17, 32, 123] (corresponding to $\tau \rightarrow 2\,\tau$), as can be deduced from the identities on these two pullbacks:

$$\frac{27\,x^2 \cdot (1-x)^2\,(1+8\,x)}{4\,(1+7\,x+x^2)^3} \;=\; \frac{1728x^2}{(x+256)^3} \circ \frac{16\,(1+8\,x)}{(x-1)\,x}, \qquad \text{(M.6)}$$

$$-\frac{108\,x \cdot (1-x)\,(1+8\,x)^2}{(1-4\,x)^6} \;=\; \frac{1728x}{(x+16)^3} \circ \frac{16\,(1+8\,x)}{(x-1)\,x}. \qquad \text{(M.7)}$$

The relation between the hypergeometric function ${}_2F_1([1/6, 1/6], [1], x)$ and ${}_2F_1([1/12, 5/12], [1], x)$, can also be understood from the Kummers's quadratic relation on ${}_2F_1([1/6, 5/6], [1], x)$:

$${}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], x\Big) \;=\; {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 4\,x \cdot (1-x)\Big).$$

Appendix N. Calabi-Yau condition versus integrality: a Saalschutzian hypergeometric family of operators

The Calabi-Yau condition that the exterior square of an order-four linear differential operator is of order five is a fundamental condition defining Calabi-Yau ODEs [88, 90, 91]. Let us consider the following *Saalschutzian hypergeometric function*

$${}_4F_3([b-c+d,\, b,\, c,\, d],\, [e,\, b+d,\, 1+b+d-e],\, x), \qquad \text{(N.1)}$$

and, let us introduce the order-four linear differential operator $M_4(b, c, d, e)$ which annihilates this hypergeometric function. It is a straightforward exercise to verify that this linear differential operator satisfies the Calabi-Yau condition: *its exterior square of is actually of order five for any values of the parameters* b, c, d, e. This order-four operator is almost self-adjoint†:

$$x^{b+d-1} \cdot M_4(b, c, d, e) \;=\; adjoint(M_4(b, c, d, e)) \cdot x^{b+d-1}. \qquad \text{(N.2)}$$

† If one denotes by N the denominator of the rational number $b+d-1$, the symmetric N-th power of $M_4(b, c, d, e)$ is up to a normalisation, self-adjoint.

For generic rational parameters, the series expansion of the *Saalschutzian* hypergeometric function (N.1) is *not globally bounded*.

Let us restrict to the condition $d = 1 - b$:

$$ {}_4F_3([1 - c, b, c, 1 - b], [e, 1, 2 - e], x). \tag{N.3}$$

The order-four operator is still such that its exterior square is of order five, but it is now a *self-adjoint operator* (see (N.2)). Again, one sees that, for generic rational parameters, the series expansion of the Saalschutzian hypergeometric function (N.3) is *not globally bounded*. Actually the four solutions are not of a MUM form. Besides the analytic at $x = 0$, non globally bounded, solution (N.3)

$$ 1 \quad + \frac{b\,c\,d \cdot (b - c + d)}{e \cdot (b + d) \cdot (1 + b + d - e)} \cdot x \quad + \quad \cdots, \tag{N.4}$$

the three other solutions have *no logarithm* and are, for generic rational parameters, *Puiseux series*:

$$ x^{1-e} \cdot \Bigl(1 \quad - \frac{(1 + d - e)\,(1 + c - e)\,(1 + b - e)\,(1 + b - c + d - e)}{(e - 2)\,(1 + b + d - e)\,(b + d + 2 - 2\,e)} \cdot x \; + \cdots\Bigr), $$
$$ x^{1-b-d} \cdot (1 \; + \; \cdots), \qquad\qquad x^{e-b-d} \cdot (1 \; + \; \cdots). $$

When the exponents $1 - e$, $1 - b - d$, $e - b - d$ are integer values one can recover solutions with logarithms for the operators.

Appendix O. Integrality of a one-parameter Saalschutzian hypergeometric family of operators

Let us consider the order-four linear differential operator ($\theta = x \cdot D_x$):

$$ M_4(\mu) \quad = \quad 16 \cdot \theta^2 \cdot (\theta - 1)^2 \\ \qquad - x \cdot (2\,\theta + 1)^2 \cdot (2\,\theta - 1 + \mu) \cdot (2\,\theta - 1 - \mu). \tag{O.1}$$

We have a one-parameter family of operators depending on μ^2.

The Wronskian of $M_4(\mu)$ is independent of μ, and reads $1/(x - 1)^2/x^4$. This order-four operator is, *for any rational value of* μ, non-trivially homomorphic to its adjoint (with order-two intertwiners):

$$ adjoint(M_4(\mu)) \cdot (x - 1) \cdot L_2(\mu) \quad = \quad L_2(\mu) \cdot (x - 1) \cdot M_4(\mu), \tag{O.2}$$

$$ M_4(\mu) \cdot (x - 1) \cdot M_2(\mu) \quad = \quad M_2(\mu) \cdot (x - 1) \cdot adjoint(M_4(\mu)), \tag{O.3}$$

where $L_2(\mu)$ and $M_2(\mu)$ are two *self-adjoint order-two* linear differential operators:

$$ L_2(\mu) \quad = \quad D_x^2 \quad - \frac{1 - \mu^2}{4\,\mu^2} \cdot \frac{1 - \mu^2\,x}{(x - 1) \cdot x^2}, \tag{O.4}$$

$$ M_2(\mu) \quad = \quad 16 \cdot x^4 \cdot D_x^2 \quad + 64 \cdot x^3 \cdot D_x \; + \; 36\,x^2 + \frac{4\,x^2}{(x - 1) \cdot \mu^2}. $$

The order-two operator $M_2(\mu)$ has simple hypergeometric solutions:

$$ (1 - x) \cdot x^{-(3\,\mu + 1)/2/\mu} \cdot {}_2F_1\Bigl([\frac{2\,\mu - 1}{2\,\mu}, \frac{2\,\mu - 1}{2\,\mu}], [\frac{\mu - 1}{\mu}], x\Bigr), $$
$$ (1 - x) \cdot x^{-(3\,\mu - 1)/2/\mu} \cdot {}_2F_1\Bigl([\frac{2\,\mu + 1}{2\,\mu}, \frac{2\,\mu + 1}{2\,\mu}], [\frac{\mu + 1}{\mu}], x\Bigr). $$

note that, generically, the hypergeometric function ${}_2F_1([\frac{2\mu-1}{2\mu}, \frac{2\mu-1}{2\mu}], [\frac{\mu-1}{\mu}], x)$ *does not corresponds to a globally bounded series*, as can be seen, for instance, for $\mu = -5/11$.

The two intertwining relations (O.2) and (O.3) can, in fact, be seen as a straight consequence of the fact that the order-four operator $M_4(\mu)$ can remarkably be written in terms of these two self-dual operators:

$$M_4(\mu) \quad = \quad M_2(\mu) \cdot (1-x) \cdot L_2(\mu) \quad + \quad \frac{(\mu^2-1)^2}{\mu^4 \cdot (1-x)}. \tag{O.5}$$

This order-four operator (O.1), or (O.5), *does not satisfy the Calabi-Yau condition*. Its exterior square $\mathcal{M}_6(\mu)$ is of *order six*, with a, not only *rational function* solution, but a *constant solution*. It is actually the direct sum of D_x and of an order-five linear differential operator $\mathcal{M}_5(\mu)$:

$$\mathcal{M}_6(\mu) \quad = \quad D_x \oplus \mathcal{M}_5(\mu), \qquad \text{where:} \tag{O.6}$$

$$\mathcal{M}_5(\mu) \quad = \quad \mathcal{M}_5(0) + \mu^2 \cdot \mathcal{M}_5^{(2)} + \mu^4 \cdot \mathcal{M}_5^{(4)} + \mu^6 \cdot \mathcal{M}_2^{(6)} \cdot D_x + \mu^8 \cdot x^3 \cdot D_x.$$

where the $\mathcal{M}_n^{(m)}$'s are linear differential operators of order n. For $\mu^2 = 1$ the order-five operator $\mathcal{M}_5(\mu)$ becomes the product‡ of two order-two operators and of D_x:

$$\mathcal{M}_5(\pm 1) \quad = \quad \mathcal{N}_2 \cdot \mathcal{M}_2 \cdot D_x. \tag{O.7}$$

For the other odd values of μ ($\mu = \pm 3, \pm 5, \cdots$) the order-five operator $\mathcal{M}_5(\mu)$ factorizes into the product of an order-two, order-one and an order-two operator.

As will be seen in a forthcoming publication, the fact that the exterior square has a *rational solution* is also a consequence of the decomposition (O.5). We have the following general result. Any order-four linear differential operator of the form

$$M_4 \quad = \quad M_2 \cdot c_0(x) \cdot L_2 \quad + \quad \frac{\lambda}{c_0(x)}, \tag{O.8}$$

where L_2 and M_2 are two (general) self-adjoint operators

$$L_2 \quad = \quad a_2(x) \cdot D_x^2 \; + \; \frac{d\,a_2(x)}{dx} \cdot D_x \; + \; a_0(x), \tag{O.9}$$

$$M_2 \quad = \quad b_2(x) \cdot D_x^2 \; + \; \frac{d\,b_2(x)}{dx} \cdot D_x \; + \; b_0(x), \tag{O.10}$$

is such that its exterior square has $A/a_2(x)$ (A is any constant) as a solution. In the case (O.5), the solution is the constant solution: $1/a_2(x) = 1$. Instead of M_4, we can introduce

$$\tilde{M}_4 \; = \; c_0(x) \cdot M_4 \quad = \quad c_0(x) \cdot M_2 \cdot c_0(x) \cdot L_2 \quad + \quad \lambda, \tag{O.11}$$

which annihilates the same solutions as M_4. It is worth noting that a decomposition like (O.8), provides, in fact, interesting results on the spectrum of $\tilde{M}_4$. For simplicity let us consider the example (O.1) with its decomposition (O.5), or on $\tilde{M}_4(\mu)$, the decomposition

$$\tilde{M}_4(\mu) \; = \quad (1-x) \cdot M_4 \quad = \quad \hat{M}_4(\mu) \quad + \quad \frac{(\mu^2-1)^2}{\mu^4}, \tag{O.12}$$

where the order-four operator $\hat{M}_4(\mu)$ factors:

$$\hat{M}_4(\mu) \quad = \quad (1-x) \cdot M_2(\mu) \cdot (1-x) \cdot L_2(\mu). \tag{O.13}$$

‡ It is also a direct sum $\hat{M}_2 \oplus (\hat{N}_2 \cdot D_x)$ where $\hat{M}_2$ and $\hat{N}_2$ are two order-two operators.

This order-four operator $\hat{M}_4(\mu)$ is, also, (non-trivially) *homomorphic to its adjoint*:

$$adjoint(\hat{M}_4(\mu)) \cdot L_2(\mu) \quad = \quad L_2(\mu) \cdot \hat{M}_4(\mu). \tag{O.14}$$

It is clear that $\mu = \pm 1$ needs to be analysed separately. For instance, for $\mu = 1$, the order-four operator $\hat{M}_4(\mu)$ factors in a direct sum $D_x \oplus M_3$ where the order-three operator M_3 has the three solutions:

$$9\,x^2 \cdot {}_4F_3\Big([1,\,1,\,\frac{5}{2},\,\frac{5}{2}],\,[2,3,3],x\Big) \,+\, 16\,x \cdot \ln(x) \,-\, 64, \tag{O.15}$$
$$x \cdot \int \frac{\pi}{2} \cdot {}_2F_1\Big([\frac{1}{2},\frac{1}{2}],\,[1],\,1-x\Big) \cdot \frac{dx}{x^2}, \qquad \text{and:} \qquad x.$$

Let us consider the solutions of $L_2(\mu)$ which can actually be expressed in terms of hypergeometric functions:

$$\Psi_1 \;=\; (1-x) \cdot x^{(2\mu-\rho)/4/\mu} \cdot {}_2F_1\Big([\frac{2\mu^2+4\mu-\rho}{4\mu},\,\frac{2\mu^2-4\mu+\rho}{4\mu}],\,[\frac{2\mu-\rho}{2\mu}],\,x\Big),$$
$$\Psi_2 \;=\; (1-x) \cdot x^{(2\mu+\rho)/4/\mu} \cdot {}_2F_1\Big([\frac{2\mu^2+4\mu+\rho}{4\mu},\,\frac{-2\mu^2+4\mu-\rho}{4\mu}],\,[\frac{2\mu+\rho}{2\mu}],\,x\Big),$$

where ρ reads $\rho = 2 \cdot (2\mu^2-1)^{1/2}$. One immediately deduces from the decomposition (O.12), that Ψ_1 and Ψ_1 are *two eigenfunctions*† *of the order-four operator* $\tilde{M}_4(\mu)$, with the *same eigenvalue* $(\mu^2-1)^2/\mu^4$:

$$\tilde{M}_4(\mu) \cdot \Psi_i \;=\; \frac{(\mu^2-1)^2}{\mu^4} \cdot \Psi_i, \qquad\qquad i = 1,\,2. \tag{O.16}$$

Introducing a rational parametrisation for μ and ρ, namely

$$\mu \,=\, \frac{u^2-2\,u+2}{u^2-2}, \qquad\qquad \rho \,=\, -2 \cdot \frac{u^2-4\,u+2}{u^2-2}, \tag{O.17}$$

one finds that, even for rational values of the parameter u, that forces μ and ρ *to be rational numbers* as well, the hypergeometric functions in the two previous eigenfunctions Ψ_i *do not correspond* (generically) to *globally bounded series*. More generally, for rational values of the parameter u, the four solutions of $\hat{M}_4(\mu)$ are Puiseux series of the form $x^r \cdot A(x)$, where r is a rational number and $A(x)$ are series analytic at $x = 0$. None of the four $A(x)$ corresponds to a globally bounded series.

Appendix O.1. Saalschutzian hypergeometric solution

In fact the order-four linear differential operator $M_4(\mu)$, or $\tilde{M}_4(\mu)$, has the ${}_4F_3$ (Saalschutzian) hypergeometric solution

$$x \cdot {}_4F_3\Big([\frac{1-\mu}{2},\,\frac{1+\mu}{2},\,\frac{3}{2},\,\frac{3}{2}],\,[1,2,2],\,x\Big), \tag{O.18}$$

which expands as:

$$x \;-\frac{9}{64}\,(\mu-1)\,(\mu+1) \cdot x^2 \;+\frac{25}{4096}\,(\mu-1)\,(\mu+1)\,(\mu-3)\,(\mu+3) \cdot x^3$$
$$-\,\frac{1225}{9437184}\,(\mu-1)\,(\mu+1)\,(\mu-3)\,(\mu+3)\,(\mu-5)\,(\mu+5) \cdot x^4 \tag{O.19}$$
$$+\,\frac{441}{268435456}\,(\mu-1)\,(\mu+1)\,(\mu-3)\,(\mu+3)\,(\mu-5)\,(\mu+5)\,(\mu-7)\,(\mu+7) \cdot x^5 \;+\; \cdots$$

† See the spectral theory of ordinary linear differential equations, in particular for self-adjoint operators.

On this expansion it is clear that the hypergeometric function truncates into a polynomial for μ *any odd integer* (positive or negative). Furthermore, this series (O.19) is globally bounded for *any rational number* μ.

Let us take a simple rational value for μ, namely $\mu = 1/3$. The series expansion (O.19) reads

$$x + \frac{1}{8}x^2 + \frac{125}{2592}x^3 + \frac{42875}{1679616}x^4 + \frac{94325}{5971968}x^5 + \frac{41544503}{3869835264}x^6 + \cdots \quad \text{(O.20)}$$

The series expansion of the nome (as well as the mirror map) is *not globally bounded*:

$$q = x + \frac{17}{48}x^2 + \frac{22195}{124416}x^3 + \frac{1913687}{17915904}x^4 + \frac{2195016283}{30958682112}x^5 + \cdots$$

In contrast, the series expansion of the Yukawa coupling $K(x)$ is a *globally bounded series*

$$K(x) = \frac{1}{x} - \frac{43}{144} - \frac{11}{288}x - \frac{31517}{3359232}x^2 - \frac{522821}{3869835264}x^3 + \cdots$$

Actually the rescaling $x \rightarrow 2^4 \cdot 3^3 \cdot x$ changes $2^4 \cdot 3^3 \cdot K(x)$ into a series with *integer coefficients*:

$$\begin{aligned} 2^4 \cdot 3^3 \cdot K(2^4 \cdot 3^3 \cdot x) &= \frac{1}{x} - 129 - 7128\,x - 756408\,x^2 - 4705389\,x^3 \\ &+ 58331013489\,x^4 + 38259799407522\,x^5 + 19576957591348938\,x^6 \qquad \text{(O.21)} \\ &+ 9193736880930978297\,x^7 + 4149261387452007788523\,x^8 + \cdots \end{aligned}$$

Appendix O.2. A non-trivially equivalent operator for $\mu = \pm 1$

Let us introduce the order-four operator

$$\begin{aligned} N_4 \,\, = \,\, & 16 \cdot \theta \cdot (\theta - 2) \cdot (\theta - 1)^2 \\ & - 4 \cdot x \cdot \theta \cdot (\theta - 1) \cdot (2\,\theta - 1)^2. \end{aligned} \qquad \text{(O.22)}$$

which is nothing but $\mathcal{M}_2 \cdot D_x^2$ (see (O.7)).

This operator is non-trivially homomorphic to $M_4(\pm 1)$, that is (O.1) for $\mu = \pm 1$:

$$N_4 \cdot A_2 \,\, = \,\, \Big(\frac{x}{1-x}\Big) \cdot A_2 \cdot \Big(\frac{1-x}{x}\Big) \cdot M_4(\pm 1), \qquad \text{(O.23)}$$

$$\text{where:} \qquad A_2 \,\, = \,\, (1-x) \cdot \theta \cdot (\theta - 1).$$

Operator N_4 *does verify the Calabi-Yau condition*: its exterior square is of *order five*.

The operator N_4 can also be written as a direct sum:

$$D_x^2 \,\oplus\, \Big(D_x^2 \,-\, \frac{1}{4\,x \cdot (1-x)}\Big). \qquad \text{(O.24)}$$

or, in terms of $\theta = x \cdot D_x$, the direct sum:

$$\theta \,\oplus\, (\theta - 1) \,\oplus\, \Big(16 \cdot \theta \cdot (\theta - 1) \,-\, 4\,x \cdot (2\theta - 1)^2\Big). \qquad \text{(O.25)}$$

Thus, besides the constant solution and $y(x) = x$, its solutions can simply be written in terms of hypergeometric functions, for instance

$$\begin{aligned} & x \cdot (1 - x) \cdot {}_2F_1\Big([\frac{3}{2}, \frac{3}{2}], [2]; x\Big) \qquad \text{(O.26)} \\ & \quad = \,\, 4\,x \cdot (1 - x) \cdot \frac{d}{dx}\Big({}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1]; x\Big)\Big) \end{aligned}$$

$$= \quad x \cdot (1-x)\,(1-2x)^{-3/2} \cdot {}_2F_1\Big([\frac{3}{4}, \frac{5}{4}], [2]; \, -\frac{4x \cdot (1-x)}{(1-2x)^2}\Big).$$

This series is *globally bounded*. Changing $x \rightarrow 16\,x$ turns this series into a series with *integer coefficients*:

$$16\,x \;+ 32\,x^2 \;+ 192\,x^3 + 1600\,x^4 + 15680\,x^5 + 169344\,x^6 + 1951488\,x^7 + 23557248\,x^8 + 294465600\,x^9 \;+\; \cdots \tag{O.27}$$

Appendix O.3. A family of non-trivially equivalent operators for odd integer μ

It is tempting to try to generalise (O.24), or (O.25), with a simple μ^2-ansatz:

$$D_x^2 \,\oplus\, \mathcal{M}_2(\mu) \qquad \text{where:} \qquad \mathcal{M}_2(\mu) \;=\; D_x^2 \;-\; \frac{\mu^2}{4\,x \cdot (1-x)}, \tag{O.28}$$

or, in terms of $\theta \,=\, x \cdot D_x$, the direct sum:

$$\theta \oplus (\theta - 1) \oplus \Big(16 \cdot \theta \cdot (\theta-1) \,-\, 4\,x \cdot (4\theta^2 - 4\theta + \mu^2)\Big). \tag{O.29}$$

In order to compare this ansatz operator with the initial operator (O.1), we slightly rewrite it as

$$16 \cdot \theta^2 \cdot (\theta-1)^2 \;-\, x \cdot (2\,\theta+1)^2 \cdot (4\theta^2 - 4\theta + 1 \,-\, \mu^2),$$

and we, also, rewrite this ansatz operator as follows:

$$\mathcal{C}_{odd}(\mu) \;=\; 16 \cdot \theta \cdot (\theta-1)^2 \cdot (\theta-2) \;-\; 4 \cdot x \cdot \theta \cdot (\theta-1) \cdot (4\theta^2 - 4\theta + \mu^2). \tag{O.30}$$

One finds that such an ansatz is actually non-trivially† *homomorphic to operator* (O.1) for *any odd integer values* (positive or negative) of μ, and that its exterior square is actually of *order-five* for *any value* of μ (Calabi-Yau condition).

For $\mu \,=\, 2\,t/(1+t^2)$, with t a *rational number* (hence $|\mu| \,<\, 1$, the solutions of (O.30) read (besides the constant solution and $y(x) \,=\, x$) simple hypergeometric functions, for instance:

$$\mathcal{S}_0(x) \;=\; x \cdot {}_2F_1\Big([\frac{t^2}{1+t^2}, \frac{1}{1+t^2}], [2], x\Big), \tag{O.31}$$

together with the solution§ $\mathcal{S}_1(x) \,=\, \mathcal{S}_0(x) \cdot \ln(x) + \tilde{\mathcal{S}}_1(x)$ where $\tilde{\mathcal{S}}_1(x)$ is analytic at $x \,=\, 0$, and solution of an order-four operator $\mathcal{N}_4$, product of two order-two operators, $\mathcal{N}_4 \,=\, \mathcal{N}_2(\mu) \cdot \mathcal{M}_2(\mu)$, where $\mathcal{N}_2(\mu)$ is an order-two operator homomorphic to $\mathcal{M}_2(\mu)$ (see (O.28)):

$$\mathcal{N}_2(\mu) \cdot \frac{1}{x^2} \cdot (2\,\theta - 1) \;=\; \frac{2}{x} \cdot \Big(D_x - \frac{d\ln(\rho(x))}{dx}\Big) \cdot \mathcal{M}_2(\mu), \tag{O.32}$$

$$\text{where:} \qquad \rho(x) \;=\; \frac{1 - x + \mu^2\,x}{(1-x) \cdot x^{3/2}}.$$

The series expansion of (O.31) is *globally bounded for any rational value*¶ of the parameter t:

$$x \;+ \frac{1}{2}\,\frac{t^2}{(1+t^2)^2} \cdot x^2 \;+ \frac{1}{12}\,\frac{t^2\,(2\,t^2+1)\,(t^2+2)}{(1+t^2)^4} \cdot x^3 + \frac{1}{144}\,\frac{t^2\,(3\,t^2+2)\,(2\,t^2+3)\,(2\,t^2+1)\,(t^2+2)}{(1+t^2)^6} \cdot x^4 \;+\; \cdots \tag{O.33}$$

† The intertwiners are linear differential operators of *order three*.
§ This solution can also be written $MeijerG([[],[(2+t^2)/(1+t^2),(2\,t^2+1)/(1+t^2)]],[[0,1],[]],x)$, i.e. as a MeijerG function.
¶ In contrast the series $\tilde{\mathcal{S}}_1(x)$ is *not globally bounded* for the rational values of t.

For instance for $\mu = 4/5$ the hypergeometric solution (O.31) reads $\mathcal{S}_0(x) = x \cdot {}_2F_1([1/5, 4/5], [2], x)$, corresponding to the *globally bounded* solution series:

$$x + \frac{2}{25}x^2 + \frac{18}{625}x^3 + \frac{231}{15625}x^4 + \frac{17556}{1953125}x^5 + \frac{1474704}{244140625}x^6 + \cdots$$

The rescaling $x \to 5^3 x$ turns this series into a series with *integer coefficients*.

Remark that the two solutions $\mathcal{S}_0(x)$ and $\mathcal{S}_1(x)$, for $\mu = 4/5$, can be replaced by the two solutions well-suited for x large ($z = 1/x$):

$$z^{-4/5} \cdot (1-z) \cdot {}_2F_1\Big([\frac{1}{5}, \frac{6}{5}], [\frac{2}{5}], z\Big), \qquad z^{-1/5} \cdot (1-z) \cdot {}_2F_1\Big([\frac{4}{5}, \frac{9}{5}], [\frac{8}{5}], z\Big).$$

The two hypergeometric functions ${}_2F_1([1/5, 6/5], [2/5], z)$ and ${}_2F_1([4/5, /5], [8/5], z)$ do not correspond to globally bounded series. We have a similar result for the other rational values of t.

Do note, however, that for the other rational values of μ we have a drastically different situation: the operator (O.30) is *no longer globally nilpotent*‡, as can be seen on the solution of $D_x^2 - \mu^2/4/x/(1-x)$ in (O.28)

$$x \cdot (1-x) \cdot {}_2F_1\Big(\Big[\frac{3 + (1-\mu^2)^{1/2}}{2}, \frac{3 - (1-\mu^2)^{1/2}}{2}\Big], [2], x\Big), \qquad \text{(O.34)}$$

which has the series expansion

$$x + \frac{1}{8}\mu^2 \cdot x^2 + \frac{1}{192}\mu^2 \cdot (\mu^2+8) \cdot x^3 + \frac{1}{9216}\mu^2 \cdot (\mu^2+8)(\mu^2+24) \cdot x^4$$
$$+ \frac{1}{737280}\mu^2 \cdot (\mu^2+8)(\mu^2+24)(\mu^2+48) \cdot x^5 + \cdots \qquad \text{(O.35)}$$

For rational values of μ that are not of the form $\mu = 2t/(1+t^2)$, the solution-series (O.35) of the operator (O.30) is *not globally bounded* as can be checked with the two rational values of μ such that $|\mu| < 1$ and $|\mu| > 1$ respectively. One can verify that the solution-series (O.35) of (O.30) for $\mu = 1/3$

$$x + \frac{1}{72}x^2 + \frac{73}{15552}x^3 + \frac{15841}{6718464}x^4 + \frac{6859153}{4837294080}x^5 + \frac{4945449313}{5224277606400}x^6 + \cdots$$

and for $\mu = 3$:

$$x + \frac{9}{8}x^2 + \frac{51}{64}x^3 + \frac{561}{1024}x^4 + \frac{31977}{81920}x^5 + \frac{948651}{3276800}x^6 + \frac{40791993}{183500800}x^7 + \cdots$$

are *not globally bounded*.

Appendix O.4. Seeking for equivalent operators for other values of μ

The operator (O.30), which is non-trivially homomorphic to (O.1), (or (O.30)) for *odd integer values* of the parameter μ, is *not valid for even integer values* of μ. For example, for $\mu = 0$, operator (O.1) actually verifies the Calabi-Yau condition, its exterior square being nothing but $\mathcal{M}_5(\mu)$ for $\mu = 0$ (see (O.6)). Operator $\mathcal{M}_5(0)$ reads

$$16 \cdot \theta^2 \cdot (\theta-1)^2 - x \cdot (2\theta+1)^2 \cdot (2\theta-1)^2. \qquad \text{(O.36)}$$

It is not of the form (O.30), which cannot encapsulate all the operators homomorphic to (O.30) satisfying the Calabi-Yau condition. As previously remarked, the series

‡ For instance one sees explicitly on (O.34) that its exponents are *not rational numbers*, therefore operator (O.30) *cannot be a globally nilpotent operator*.

expansion of the solution $x \cdot {}_4F_3\Big([\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}], [1,2,2], x)$ is globally bounded, and can be turned into a series with integer coefficients with the rescaling $x \rightarrow 256\, x$.

Note that (O.36) can be used as a "seed" to get a family of μ^2-dependent operator verifying the Calabi-Yau condition. The linear differential operator

$$\mathcal{C}(\mu) \,\,=\,\, 16 \cdot \theta^2 \cdot (\theta\,-1)^2 \qquad \qquad (O.37)$$
$$- \, x \cdot (2\theta\, +1\,-\mu) \cdot (2\theta\, +1 + \mu) \cdot (2\theta\, -1\,-\mu) \cdot (2\theta\, -1 + \mu),$$

is such that *its exterior square is actually of order five*. Very fortunately, operator (O.37) is *non-trivially homomorphic* to (O.1) for *any integer value* (positive or negative, even or odd) of parameter μ. Again the intertwiners are of order-three.

For instance, for $\mu\,=\,2$ one has the intertwining relation between operator (O.1) and (O.37)

$$\mathcal{C}(2) \cdot \Big((8\,\theta^2 \cdot (2\,\theta\,-3) \,+\, 4\,\theta\,+1) \,-\, 2\,x \cdot (2\,\theta\,-3)\,(2\,\theta\,+1)^2\Big) \qquad (O.38)$$
$$= \,\Big((8\,\theta^2 \cdot (2\,\theta\,-3) \,+\, 4\,\theta\,+1) \,-\, 2\,x \cdot (2\,\theta\,-3)\,(2\,\theta\,+1)\,(2\,\theta\,+3)\Big) \cdot M_4(2).$$

For larger values of the integer μ the intertwiners become more and more involved. They are still of degree three in θ but of higher degree in x.

Operator (O.37) has simple hypergeometric solutions for *any value* of μ:

$$x \cdot {}_4F_3\Big([\frac{\mu\,+3}{2}, \,\frac{-\mu\,+3}{2}, \,\frac{\mu\,+1}{2}, \,\frac{-\mu\,+2}{2}], \,[1,\,2,\,2]; \,\,x) \qquad (O.39)$$

For *any rational value* of μ, the series expansion of the solution of (O.37) analytic at $x\,=\,0$, namely (O.39)

$$x \,\,+\frac{1}{64}\,(\mu-1)\,(\mu+1)\,(\mu-3) \cdot (\mu+3)\,x^2 \qquad (O.40)$$
$$+\,\frac{1}{36864}\,(\mu-1)\,(\mu+1)\,(\mu-3)^2\,(\mu+3)^2\,(\mu-5)\,(\mu+5)\,\cdot\,x^3 \,\,+\,\cdots$$

is *globally bounded*†.

For instance, for $\mu\,=\,1/3$, the series (O.40)

$$x + \frac{10}{81}\,x^2 + \frac{2800}{59049}\,x^3 + \frac{1078000}{43046721}\,x^4 + \frac{53953900}{3486784401}\,x^5 + \frac{26709338656}{2541865828329}\,x^6 \,+\,\cdots$$

can be turned into a series with integer coefficients after the rescaling $x\,\rightarrow\,3^6\,x$. However, the series expansion of the *nome* for (O.37) for $\mu\,=\,1/3$ is *not globally bounded*:

$$x \,+\frac{19}{54}\,x^2 \,+\frac{250403}{1417176}\,x^3 \,+\frac{218211473}{2066242608}\,x^4 \,+\frac{281241377443}{4016775629952}\,x^5$$
$$+\,\frac{1456188325082179}{29282294342350080}\,x^6 \,+\frac{3167628271177596809}{85387170302292833280}\,x^7 \,+\,\cdots$$

In contrast, the Yukawa coupling $K(x)$ for $\mu\,=\,1/3$ is *actually globally bounded*

$$K(x) \,\,=\,\, \frac{1}{x}\,-\frac{49}{162}\,-\frac{2179}{59049}\,x\,-\frac{1508129}{172186884}\,x^2 + \frac{47590097}{167365651248}\,x^3 \,+\,\cdots$$

Actually the rescaling $x\,\rightarrow\,2 \cdot 3^6\,x$ on $2 \cdot 3^6\,K(x)$ turns the previous series expansion into a series with *integer coefficients*:

$$2 \cdot 3^6 \cdot K(2 \cdot 3^6\,x) \,\,=\,\, \frac{1}{x}\,-441\,-78444\,x\,-27146322\,x^2\,+1284932619\,x^3$$
$$+\,27674475754905\,x^4 + 591191131097467 98\,x^5 + 1008960414836939 39736\,x^6$$
$$+\,158984355721045048019613\,x^7\,+241323001023828827752150059\,x^8\,+\,\cdots$$

† Operator (O.37) is thus globally nilpotent for any rational value of μ.

to be compared with (O.21) for operator (O.1). We have similar result for $\mu = 4/5$ (the series (O.40) can be turned into a series with integer coefficients after the rescaling $x \rightarrow 2^8 \cdot 5^5 x$, and the series expansion of the nome is not globally bounded). The Yukawa coupling $K(x)$ for $\mu = 4/5$ can be changed into a series with *integer coefficients*:

$$2^8 \cdot 3 \cdot 5^5 \, K(2^8 \cdot 3 \cdot 5^5 \, x) \; = \; \frac{1}{x} \; - 1057980 \; - 92574954000 \, x \qquad \text{(O.41)}$$
$$- \, 51733629839745000 \, x^2 \, + 745090920366867786859 20 \, x^3 \, + \, \cdots$$

Note that, for *any odd integer*, (positive or negative), (O.40) is a *terminating series*, reducing to a *polynomial*.

As a byproduct, we see, for *any odd integer* value of μ, that operator (O.1) is (non-trivially) homomorphic to *two different* operators (O.30) and (O.37), such that their exterior square is of order five. As it should these two operators (O.30) and (O.37) are, for *any odd integer value* of μ, homomorphic† (with intertwiners of order three). For $\mu = 3$ one gets the homomorphism:

$$\mathcal{C}_{odd}(3) \cdot \Big(4\,\theta \cdot (\theta-1)^2 \, - 4\,x \cdot (\theta-1) \cdot (\theta-2) \cdot (\theta+2)\Big) \qquad \text{(O.42)}$$
$$= \quad \Big(4\,(\theta-2) \cdot (\theta-1)^2 \, - \, x \cdot (\theta-1) \cdot (4\,\theta^2 \, - 4\,\theta \, + \, 9)\Big) \cdot \mathcal{C}(3).$$

For other values of μ one gets slightly more involved intertwiners that are no longer of degree one in x.

The homomorphisms, for *odd integer values* of μ, between a non globally nilpotent operator (O.30) and the globally nilpotent operator (O.37), or between operators with non globally bounded and globally bounded solutions, may seem misleading. It is important to note that, for *odd integer values* of μ, the operator (O.37) is *non longer irreducible*‡, that the intertwiners between (O.30) and (O.37) (see (O.42)) are not globally nilpotent, and, furthermore, that the globally bounded infinite series (O.40), reduces to a polynomial. The intertwining relation between (O.30) and (O.37) then matches this terminating hypergeometric series (O.40) with the $y(x) = x$ solution of (O.30).

Do note that, given an order-four operator such that its *exterior square has a rational solution* ("extended Calabi-Yau condition"), finding an order-four operator (non-trivially) homomorphic to the first operator such that its *exterior square is of order five* (Calabi-Yau condition) is an extremely difficult task¶, even if one assumes decompositions like (O.8), (O.11). We will address this difficult question of the reduction of the "extended Calabi-Yau condition" to the "Calabi-Yau condition" in a forthcoming publication.

† Note that for even integer values of μ, operator (O.37) is irreducible when (O.30) is reducible. Therefore operators (O.30) and (O.37) *cannot be homomorphic* for *even integer* values of μ.

‡ For any odd integer value of μ the operator (O.37) is the product of four order-one operators such that their wronskian is a rational function. Operator (O.37) is thus globally nilpotent.

¶ In particular because, as we have seen, this reduction to an operator satisfying the Calabi-Yau condition is not unique. In contrast, starting from an operator satisfying the Calabi-Yau condition, like (O.37), it is straightforward to get operators that are non-trivially homomorphic to this operator, and are such that their exterior square has a rational solution (just perform the LCLM of this operator with any order-three linear differential operator). This will be explained in a forthcoming publication.

Appendix P. Modular forms and selected ${}_2F_1$ hypergeometric functions with two pullbacks

We display here a (non exhaustive) list of miscellaneous identities (between modular forms and their representations as ${}_2F_1$ hypergeometric functions with two pullbacks) that we often encountered in our studies of the Ising model, lattice Green functions, or Calabi-Yau ODEs:

$$
\begin{aligned}
&{}_2F_1\Big([\frac{1}{6}, \frac{1}{3}], [1];\, 108 \cdot x^2 \cdot (1+4\,x)\Big) \qquad \text{(P.1)} \\
&\quad = \quad (1-12\,x)^{-1/2} \cdot {}_2F_1\Big([\frac{1}{6}, \frac{1}{3}], [1];\, -\frac{108 \cdot x \cdot (1+4\,x)^2}{(1-12\,x)^3}\Big). \\
&\quad = \quad 1 \; + 6\,x^2 \; + 24\,x^3 \; + 252\,x^4 \; + 2016\,x^5 \; + 19320\,x^6 \; + 183456\,x^7 \\
&\qquad\qquad + 1823094\,x^8 \; + 18406752\,x^9 \; + 189532980\,x^{10} \; + \; \cdots
\end{aligned}
$$

$$
\begin{aligned}
&(1\,+2\,x) \cdot {}_2F_1\Big([\frac{1}{6}, \frac{1}{3}], [1];\, \frac{27\,x^2\,(1\,+x)^2}{4\,(1\,+\,x\,+x^2)^3}\Big) \qquad \text{(P.2)} \\
&\quad = \quad (1\,+\,x\,+x^2)^{1/2} \cdot {}_2F_1\Big([\frac{1}{6}, \frac{1}{3}], [1];\, \frac{27\,x\,(1\,+x)}{4\,(1\,+2\,x)^6}\Big).
\end{aligned}
$$

The series expansion of (P.2) are globally bounded and can be changed into a series with integer coefficients after the rescaling $x \rightarrow 4\,x$. This identity is nothing but identity (P.1) after a change of variable. Another example corresponds to HeunG functions of the form $HeunG(a,\, q,\, 1,\, 1,\, 1,\, 1;\, x)$, such that the two parameters a and q are associated with fixed points of the symmetries (139) of these HeunG functions :

$$
\begin{aligned}
&HeunG(\frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1, 4\,x) \quad = \quad HeunG(2,\, 1, 1, 1, 1, 1, 8\,x) \\
&\quad = \quad \frac{1}{1\,-4\,x} \cdot HeunG\Big(-1,\, 0, 1, 1, 1, 1,\, -\frac{4\,x}{1\,-4\,x}\Big) \\
&\quad = \quad \frac{1}{1\,-4\,x} \cdot {}_2F_1([\frac{1}{4}, \frac{1}{4}],\, [1],\, \frac{64\,x^2 \cdot\, (1\,-8\,x)}{(1\,-4\,x)^4}\Big) \\
&\quad = \quad {}_2F_1([\frac{1}{4}, \frac{1}{4}],\, [1],\, 64\,x \cdot (1\,-4\,x) \cdot (1\,-8\,x)^2\Big) \qquad \text{(P.3)} \\
&\quad = \quad (1\,-256\,x\,+5120\,x^2\,-32768\,x^3\,+65536\,x^4)^{-1/4} \\
&\qquad \times\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}],\, [1],\, -1728 \cdot \frac{x \cdot (1\,-4\,x) \cdot (1\,-8\,x)^2}{(1\,-256\,x\,+5120\,x^2\,-32768\,x^3\,+65536\,x^4)^3}\Big) \\
&\quad = \quad (1\,-16\,x\,+80\,x^2\,-128\,x^3\,+256\,x^4)^{-1/4} \\
&\qquad \times\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}],\, [1],\, 1728 \cdot \frac{x^4 \cdot (1\,-4\,x)^4 \cdot (1\,-8\,x)^2}{(1\,-16\,x\,+80\,x^2\,-128\,x^3\,+256\,x^4)^3}\Big) \\
&\quad = \quad 1 \; +4\,x \; +20\,x^2 \; +112\,x^3 \; +676\,x^4 \; +4304\,x^5 \; +28496\,x^6 \; +194240\,x^7 \\
&\qquad\qquad +1353508\,x^8 \; +9593104\,x^9 \; + \; \cdots
\end{aligned}
$$

The previous HeunG function can also be written

$$
\begin{aligned}
{}_2F_1\Big([\frac{1}{2},\, \frac{1}{2}],\, [1],\, 16\,x\cdot(1\,-4\,x)\Big) \quad &= \quad \frac{1}{1\,-4\,x}\cdot\, {}_2F_1\Big([\frac{1}{2},\, \frac{1}{2}],\, [1],\, \frac{16\,x^2}{(1\,-4\,x)^2}\Big) \\
&= \quad (1\,-4\,x)\cdot\,(1\,-8\,x)\cdot\, {}_2F_1\Big([\frac{3}{2},\, \frac{3}{2}],\, [2],\, 16\,x\cdot(1\,-4\,x)\Big) \\
&\qquad -\frac{2\,x}{(1-4\,x)^3}\cdot\, {}_2F_1\Big([\frac{3}{2},\, \frac{3}{2}],\, [2],\, \frac{16\,x^2}{(1\,-4\,x)^2}\Big)
\end{aligned} \tag{P.4}
$$

Using the identity

$$
\begin{aligned}
\Big(16\,x\cdot(1\,-4\,x)\Big)\circ\Big(\frac{9\,x^2}{1+4\,x+40\,x^2}\Big) \quad &= \quad 144\,\frac{x^2\cdot\,(1\,+2\,x)^2}{(1+4\,x+40\,x^2)^2} \\
&= \quad \Big(\frac{16\,x^2}{(1\,-4\,x)^2}\Big)\circ\Big(\frac{-3\,x\cdot\,(1\,+2\,x)}{(1\,-4\,x)^2}\Big)
\end{aligned} \tag{P.5}
$$

one easily deduces from (P.4) a *linear functional identity* between the *same* hypergeometric function with *three different rational pullbacks*:

$$
\begin{aligned}
&18\,x^2\cdot\,(1+4\,x+40\,x^2)^3\cdot\,(1\,-4\,x)^6\cdot\, {}_2F_1\Big([\frac{3}{2},\, \frac{3}{2}],\, [2],\, p_1(x)\Big) \\
&\quad -\,(1\,+2\,x)^7\cdot\,(1\,-4\,x)^7\cdot\, {}_2F_1\Big([\frac{3}{2},\, \frac{3}{2}],\, [2],\, p_2(x)\Big) \\
&\quad +\,(1\,+8\,x)^2\,\cdot\,(1+4\,x+40\,x^2)^3\cdot\,(1\,+2\,x)^6\cdot\, {}_2F_1\Big([\frac{3}{2},\, \frac{3}{2}],\, [2],\, p_3(x)\Big) \quad = \quad 0,
\end{aligned} \tag{P.6}
$$

where the three pullbacks read respectively:

$$
\begin{aligned}
p_1(x) \quad &= \quad 1296\,\cdot\,\frac{x^4}{(1\,+2\,x)^4} \quad = \quad \Big(\frac{16\,x^2}{(1\,-4\,x)^2}\Big)\circ\Big(\frac{9\,x^2}{1+4\,x+40\,x^2}\Big), \\
p_2(x) \quad &= \quad 144\,\cdot\,\frac{x^2\cdot\,(1\,+2\,x)^2}{(1\,+4\,x\,+40\,x^2)^2} \\
&= \quad \Big(16\,x\cdot(1\,-4\,x)\Big)\circ\Big(\frac{9\,x^2}{1+4\,x+40\,x^2}\Big), \\
p_3(x) \quad &= \quad -48\,\cdot\,\frac{x\cdot\,(1\,+2\,x)\cdot\,(1+4\,x+40\,x^2)}{(1\,-4\,x)^4} \\
&= \quad \Big(16\,x\cdot(1\,-4\,x)\Big)\circ\Big(\frac{-3\,x\cdot\,(1\,+2\,x)}{(1\,-4\,x)^2}\Big).
\end{aligned} \tag{P.7}
$$

Note that each of the three terms in (P.6) corresponds to series with *integer coefficients*, their series reading respectively:

$$
\begin{aligned}
&18\,x^2\,-216\,x^3\,+2160\,x^4\,-25344\,x^5\,+223236\,x^6\,-1810512\,x^7\,+\cdots, \\
&-1\,+14\,x\,-190\,x^2\,+2524\,x^3\,-26732\,x^4\,+270184\,x^5\,-2650164\,x^6\,+\cdots, \\
&1\,-14\,x\,+172\,x^2\,-2308\,x^3\,+24572\,x^4\,-244840\,x^5\,+2426928\,x^6\,+\cdots
\end{aligned}
$$

Another $HeunG(a,\, q,\, 1,\, 1,\, 1,\, 1;\, x)$ example, such that the two parameters a and q correspond to fixed points of the symmetries (139) of these particular HeunG functions, read:

$$HeunG\Big(\frac{1+i\,3^{1/2}}{2},\, \frac{3+i\,3^{1/2}}{6},\, 1,\, 1,\, 1,\, 1;\;\, 9\cdot \frac{3+\,i\,3^{1/2}}{2}\cdot x\Big)$$
$$= \quad {}_2F_1\Big([\frac{1}{3},\, \frac{1}{3}],\, [1];\;\, 81\cdot x\cdot (1\,-27\,x\,+243\,x^2)\Big)$$
$$= \quad \frac{1}{x}\cdot {}_2F_1\Big([\frac{1}{3},\, \frac{1}{3}],\, [1];\;\, \frac{1-27\,x\,+243\,x^2}{729\,x^3}\Big)$$
$$= \quad 1\;\, +9\,x\;\, +81\,x^2\;\, +567\,x^3 +729\,x^4 -72171\,x^5 -1764909\,x^6 -28284471\,x^7 -343842327\,x^8 -2859802119\,x^9 -2072088459\,x^{10} +523309421259\,x^{11} +13407709577211\,x^{12} +226522478442087\,x^{13}\;\, +\;\cdots, \qquad (P.8)$$

where the coefficients are actually integers, their sign having a period six. This example is nothing but revisiting (H.31) with a $x \rightarrow -81\,x$ change of variable.

Among the fixed points of the symmetries (139) the case $a\,=\,0$ for *any value* of q, corresponds to a special limit. Let us write q as $q\,=\,(1-r^2)/4$, one has

$$lim_{a\rightarrow 0}\, HeunG(a,\, \frac{1-r^2}{4},\, 1,\, 1,\, 1,\, 1;\;\, 2^4\,a\,x) \quad = \quad {}_2F_1\Big([\frac{1-r}{2},\, \frac{1+r}{2}],\, [1],\;\, 2^4\,x\Big),$$

which is, of course, a series with integer coefficients for any even integer values of r (it is a simple polynomial expression for odd integer values of r).

More identities† read:

$$HeunG(-3,\, 0,\, 1/2,\, 1,\, 1,\, 1/2;\, 12\cdot x) \quad = \quad {}_2F_1\Big([\frac{1}{6},\, \frac{1}{3}],[1];\, 108\cdot x^2\cdot (1+4\,x)\Big)$$
$$= \quad (1-12\,x)^{-1/2}\;\cdot\; {}_2F_1\Big([\frac{1}{6},\, \frac{1}{3}],[1];-\,\frac{108\cdot x\cdot (1+4\,x)^2}{(1-12\,x)^3}\Big)$$
$$= \quad 1\;\, +12\,x^2\;\, +48\,x^3\;\, +540\,x^4\;\, +4320\,x^5\;\, +42240\,x^6\;\, +403200\,x^7 +4038300\,x^8\;\, +\;\cdots,$$

$$HeunG(4,\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\frac{1}{2};\, 64\,x) \quad =$$
$$= \quad (1\,-16\,x)^{-1/2}\;\cdot\; {}_2F_1\Big([\frac{1}{6},\, \frac{1}{3}],\, [1];\, \frac{1728\,x^2}{(1\,-16\,x)^3}\Big)$$
$$= \quad (1\,-64\,x)^{-1/2}\;\cdot\; {}_2F_1\Big([\frac{1}{6},\, \frac{1}{3}],\, [1];\, -\,\frac{432\,x}{(1\,-64\,x)^3}\Big) \qquad (P.9)$$
$$= \quad 1\;\, +8\,x\;\, +192\,x^2\;\, +6656\,x^3\;\, +275968\,x^4\;\, +12644352\,x^5\;\, +616562688\,x^6 +31366053888\,x^7\;\, +1645521666048\,x^8\;\, +88371818921984\,x^9\;\, +\;\cdots,$$

$${}_3F_2\Big([\frac{1}{3},\, \frac{1}{2},\, \frac{2}{3}],\, [1,\, 1],\, -\,\frac{108\,x}{(1\,-16\,x)^3}\Big) \qquad (P.10)$$
$$= \quad \frac{1-16\,x}{1\,-4\,x}\cdot {}_3F_2\Big([\frac{1}{3},\, \frac{1}{2},\, \frac{2}{3}],\, [1,\, 1],\, \frac{108\,x^2}{(1\,-4\,x)^3}\Big)$$
$$= \quad 1\;\, -12\,x\;\, -36\,x^2\;\, -192\,x^3\;\, -1380\,x^4\;\, -11952\,x^5\;\, -116928\,x^6 -1242624\,x^7\;\, -14006628\,x^8\;\, -164954640\,x^9\;\, +\;\cdots,$$

† Not to be confused with Goursat-type identities on hypergeometric functions (see for instance section 4 in [155]). Here, the hypergeometric functions have the *same* parameters but different pullbacks.

$$ {}_2F_1\Bigl([\frac{1}{2}, \frac{1}{2}], [1], x\Bigr) \quad = \quad (1 \, - 16\,x \, + 16\,x^2)^{-1/4} \qquad \text{(P.11)} $$
$$ \times \; {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], \; -108 \cdot \frac{x \cdot (1-x)}{(1 \, - 16\,x \, + 16\,x^2)^3}\Bigr). $$

This globally bounded series becomes a series with integer coefficients after the rescaling $x \, \rightarrow \, 16\,x$.

$$ {}_2F_1\Bigl([\frac{1}{4}, \frac{3}{4}], [1], x\Bigr) \qquad \text{(P.12)} $$
$$ = \quad (1 \, + 3\,x)^{-1/4} \, \cdot \, {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], \; 27 \cdot \frac{x \cdot (1-x)^2}{(1 \, + 3\,x)^3}\Bigr). $$

This globally bounded series becomes a series with integer coefficients after the rescaling $x \, \rightarrow \, 64\,x$.

$$ {}_2F_1\Bigl([\frac{1}{3}, \frac{2}{3}], [1], x\Bigr) \qquad \text{(P.13)} $$
$$ = \quad \Bigl(\frac{9}{9 \, - 8\,x}\Bigr)^{1/4} \, \cdot \, {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], \; 64 \cdot \frac{x^3 \cdot (1-x)}{(9 \, - 8\,x)^3}\Bigr). $$

This globally bounded series becomes a series with integer coefficients after the rescaling $\, x \, \rightarrow \, 27\,x$.

A few hypergeometric functions occur in the analysis of the Yang-Baxter integrable hard-hexagon [156, 157, 158]:

$$ {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1]; \, -1728 \, \frac{x\,\bigl(1 + 11\,x - x^2\bigr)^5}{\bigl(1 - 228\,x + 494\,x^2 + 228\,x^3 + x^4\bigr)^3}\Bigr) \qquad \text{(P.14)} $$
$$ = \quad \Bigl(\frac{1 - 228\,x + 494\,x^2 + 228\,x^3 + x^4}{x^4 - 12\,x^3 + 14\,x^2 + 12\,x + 1}\Bigr)^{1/4} $$
$$ \times \; {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1]; \, -1728 \, \frac{x^5\,\bigl(1 + 11\,x - x^2\bigr)}{\bigl(x^4 - 12\,x^3 + 14\,x^2 + 12\,x + 1\bigr)^3}\Bigr), $$

and the globally bounded algebraic hypergeometric functions ${}_2F_1([1/6, \, 2/3], \, [1/2], \, x)$ ${}_2F_1([1/4, \, 3/4], \, [2/3], \, x)$:

$$ {}_2F_1([\frac{1}{6}, \frac{2}{3}], [\frac{1}{2}]; \; 27\,x) \; = \; 1 \; + 6\,x \, + 105\,x^2 \, + 2184\,x^3 \, + 48906\,x^4 \, + 1141140\,x^5 $$
$$ + \, 27335490\,x^6 \, + 666865800\,x^7 \, + 16488256905\,x^8 \; + \; \cdots \qquad \text{(P.15)} $$
$$ {}_2F_1([\frac{1}{4}, \frac{3}{4}], [\frac{2}{3}]; \; 64\,x) \; = \; 1 \; + 18\,x \, + 756\,x^2 \, + 37422\,x^3 \, + 1990170\,x^4 $$
$$ + \, 110198556\,x^5 \, + 6261870888\,x^6 \, + 362293958520\,x^7 \; + \; \cdots \qquad \text{(P.16)} $$

Note that the two second-order operators

$$ D_x^2 \; + \frac{1}{6} \cdot \, \frac{(11\,x - 3)}{x \, \cdot \, (x-1)} \, \cdot D_x \; + \frac{1}{9} \, \frac{1}{x \cdot \, (x-1)}, $$
$$ D_x^2 \; + \frac{2}{3} \cdot \, \frac{(3\,x - 1)}{x \, \cdot \, (x-1)} \, \cdot D_x \; + \frac{3}{16} \, \frac{1}{x \, \cdot \, (x-1)}, \qquad \text{(P.17)} $$

annihilating respectively the two algebraic functions ${}_2F_1([1/6, \, 2/3], \, [1/2], \, x)$ ${}_2F_1([1/4, \, 3/4], \, [2/3], \, x)$, are such that their symmetric sixth power are (non-trivially) homomorphic, the intertwiners being order-six operators.

References

[1] Butera P and Comi M 2002 An on-line library of extended high-temperature expansions of basic observables for the spin-S Ising models on two- and three-dimensional lattices *J. Statist. Phys.* **109** 311–315 URL http://arxiv.org/abs/hep-lat/0204007

[2] Campostrini M, Pelissetto A, Rossi P and Vicari E 2002 25th-order high-temperature expansion results for three-dimensional Ising-like systems on the simple-cubic lattice *Phys. Rev. E* **65** 066127 URL http://arxiv.org/abs/cond-mat/0201180

[3] Arisue H and Fujiwara T 2003 Algorithm of the finite-lattice method for high-temperature expansion of the Ising model in three dimensions *Phys. Rev. E* **67** 066109 URL http://arxiv.org/abs/hep-lat/0209002

[4] Bessis J D, Drouffe J M and Moussa P 1976 Positivity constraints for the Ising ferromagnetic model *J. Phys. A* **9** 2105 URL http://stacks.iop.org/0305-4470/9/i=12/a=015

[5] Lalin M N and Rogers M D 2007 Functional equations for Mahler measures of genus-one curves *Algebra Number Theory* **1** 87–117 URL http://arxiv.org/abs/math/0612007v3

[6] Eisenstein G 1852 Über eine allgemeine Eigenschaft der Reihen-Entwicklungen aller algebraischen Funktionen *Bericht Königl. Preuss. Akad. Wiss. Berlin* 441–443

[7] Heine E 1854 Über die Entwickelung von Wurzeln algebraischer Gleichungen in Potenzreihen *J. Reine Angew. Math.* 267–275 URL http://dx.doi.org/10.1515/crll.1854.48.267

[8] Fatou P 1906 Séries trigonométriques et séries de Taylor *Acta Math.* **30** 335–400 URL http://dx.doi.org/10.1007/BF02418579

[9] Pólya G and Szegő G 1998 *Problems and theorems in analysis. II* Classics in Mathematics (Berlin: Springer-Verlag) ISBN 3-540-63686-2 Theory of functions, zeros, polynomials, determinants, number theory, geometry

[10] Pólya G 1916 Über Potenzreihen mit ganzzahligen Koeffizienten *Math. Ann.* **77** 497–513 URL http://dx.doi.org/10.1007/BF01456965

[11] Flajolet P, Gerhold S and Salvy B 2005 On the non-holonomic character of logarithms, powers, and the nth prime function *Electron. J. Combin.* **11** Research Paper A2, 16 pp. (electronic) URL http://www.combinatorics.org/ojs/index.php/eljc/article/view/v11i2a2

[12] Carlson F 1921 Über Potenzreihen mit ganzzahligen Koeffizienten *Math. Z.* **9** 1–13 URL http://dx.doi.org/10.1007/BF01378331

[13] Pólya G 1923 Sur Les Series Entières à Coefficients Entiers *Proc. London Math. Soc.* **21** 22–38 URL http://dx.doi.org/10.1112/plms/s2-21.1.22

[14] Zenine N, Boukraa S, Hassani S and Maillard J-M 2004 The Fuchsian differential equation of the square lattice Ising model $\chi^{(3)}$ susceptibility *J. Phys. A* **37** 9651–9668 URL http://arxiv.org/abs/math-ph/0407060

[15] Boukraa S, Hassani S, Jensen I, Maillard J-M and Zenine N 2010 High-order Fuchsian equations for the square lattice Ising model: $\chi^{(6)}$ *J. Phys. A* **43** 115201, 22 URL http://arxiv.org/abs/0912.4968

[16] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 Square lattice Ising model susceptibility: connection matrices and singular behaviour of $\chi^{(3)}$ and $\chi^{(4)}$ *J. Phys. A* **38** 9439–9474 URL http://arxiv.org/abs/math-ph/0506065

[17] Boukraa S, Hassani S, Maillard J-M and Zenine N 2007 Singularities of n-fold integrals of the Ising class and the theory of elliptic curves *J. Phys. A* **40** 11713–11748 URL http://arxiv.org/abs/0706.3367

[18] Boukraa S, Hassani S, Maillard J-M, McCoy B M and Zenine N 2007 The diagonal Ising susceptibility *J. Phys. A* **40** 8219–8236 URL http://arxiv.org/abs/math-ph/0703009

[19] Boukraa S, Guttmann A J, Hassani S, Jensen I, Maillard J-M, Nickel B and Zenine N 2008 Experimental mathematics on the magnetic susceptibility of the square lattice Ising model *J. Phys. A* **41** 455202, 51 URL http://arxiv.org/abs/0808.0763

[20] Stanley R P 1980 Differentiably finite power series *European J. Combin.* **1** 175–188

[21] Lipshitz L 1989 *D*-finite power series *J. Algebra* **122** 353–373 URL http://dx.doi.org/10.1016/0021-8693(89)90222-6

[22] Bernšteĭn I N 1972 Analytic continuation of generalized functions with respect to a parameter *Funkcional. Anal. i Priložen.* **6** 26–40 URL http://www.math.tau.ac.il/~bernstei/Publication_list/publication_texts/Bern-a-cont-FAN.pdf

[23] Kashiwara M 1978 On the holonomic systems of linear differential equations. II *Invent. Math.* **49** 121–135 URL http://dx.doi.org/10.1007/BF01403082

[24] Takayama N 1992 An approach to the zero recognition problem by Buchberger algorithm *J. Symbolic Comput.* **14** 265–282 URL http://dx.doi.org/10.1016/0747-7171(92)90039-7

[25] Wu T T, McCoy B M, Tracy C A and Barouch E 1976 Spin-spin correlation functions for the

two-dimensional Ising model: Exact theory in the scaling region *Phys. Rev. B* **13** 316–374 URL http://dx.doi.org/10.1103/PhysRevB.13.316

[26] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 Ising model susceptibility: the Fuchsian differential equation for $\chi^{(4)}$ and its factorization properties *J. Phys. A* **38** 4149–4173 URL http://arxiv.org/abs/cond-mat/0502155

[27] Bostan A, Boukraa S, Guttmann A J, Hassani S, Jensen I, Maillard J-M and Zenine N 2009 High order Fuchsian equations for the square lattice Ising model: $\tilde{\chi}^{(5)}$ *J. Phys. A* **42** 275209, 32 URL http://arxiv.org/abs/0904.1601

[28] Bostan A, Boukraa S, Hassani S, Maillard J-M, Weil J-A and Zenine N 2009 Globally nilpotent differential operators and the square Ising model *J. Phys. A* **42** 125206, 50 URL http://arxiv.org/abs/0812.4931

[29] André Y 2003 Arithmetic Gevrey series and transcendence. A survey *J. Théor. Nombres Bordeaux* **15** 1–10 Les XXIIèmes Journées Arithmetiques (Lille, 2001) URL http://jtnb.cedram.org/item?id=JTNB_2003__15_1_1_0

[30] André Y 1989 *G-functions and geometry* Aspects of Mathematics, E13 (Braunschweig: Friedr. Vieweg & Sohn) ISBN 3-528-06317-3

[31] Krattenthaler C and Rivoal T 2010 On the integrality of the Taylor coefficients of mirror maps *Duke Math. J.* **151** 175–218 URL http://arxiv.org/abs/0907.2577

[32] Bostan A, Boukraa S, Hassani S, van Hoeij M, Maillard J-M, Weil J-A and Zenine N 2011 The Ising model: from elliptic curves to modular forms and Calabi-Yau equations *J. Phys. A* **44** 045204, 44 URL http://arxiv.org/abs/1007.0535

[33] Assis M, Boukraa S, Hassani S, van Hoeij M, Maillard J-M and McCoy B M 2012 Diagonal Ising susceptibility: elliptic integrals, modular forms and Calabi-Yau equations *J. Phys. A* **45** 075205, 32 URL http://arxiv.org/abs/1110.1705

[34] Yang Y and Zudilin W 2010 On Sp_4 modularity of Picard-Fuchs differential equations for Calabi-Yau threefolds *Gems in experimental mathematics* (*Contemp. Math.* vol 517) (Providence, RI: Amer. Math. Soc.) pp 381–413 URL http://arxiv.org/abs/0803.3322

[35] Lian B H and Yau S T 1996 Mirror maps, modular relations and hypergeometric series. II *Nuclear Phys. B Proc. Suppl.* **46** 248–262 *S*-duality and mirror symmetry (Trieste, 1995) URL http://arxiv.org/abs/hep-th/9507153

[36] Ford D, McKay J and Norton S 1994 More on replicable functions *Comm. Algebra* **22** 5175–5193 URL http://dx.doi.org/10.1080/00927879408825127

[37] McKay J and Sebbar A 2007 Replicable functions: an introduction *Frontiers in number theory, physics, and geometry. II* (Berlin: Springer) pp 373–386 URL http://dx.doi.org/10.1007/978-3-540-30308-4_10

[38] El Basraoui A 2005 *Modular Functions and Replicable Functions* Master's thesis University of Ottawa Canada URL http://www.ruor.uottawa.ca/fr/bitstream/handle/10393/10771/MR16975.PDF?sequence=1

[39] Livné R, Schütt M and Yui N 2010 The modularity of $K3$ surfaces with non-symplectic group actions *Math. Ann.* **348** 333–355 URL http://arxiv.org/abs/0904.1922

[40] Schütt M 2004 New examples of modular rigid Calabi-Yau threefolds *Collect. Math.* **55** 219–228 URL http://arxiv.org/abs/math/0311106

[41] Gouvêa F Q and Yui N 2011 Rigid Calabi-Yau threefolds over $\mathbb{Q}$ are modular *Expo. Math.* **29** 142–149 URL http://arxiv.org/abs/0902.1466

[42] Fontaine J M and Mazur B 1995 Geometric Galois representations *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)* Ser. Number Theory, I (Int. Press, Cambridge, MA) pp 41–78 URL https://www.dpmms.cam.ac.uk/~ty245/2010_GalRep_L24/Fontaine-Mazur.pdf

[43] Saito M H and Yui N 2001 The modularity conjecture for rigid Calabi-Yau threefolds over Q *J. Math. Kyoto Univ.* **41** 403–419 URL http://arxiv.org/abs/math/0009041

[44] van der Poorten A 1978/79 A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$ *Math. Intelligencer* **1** 195–203 An informal report URL http://dx.doi.org/10.1007/BF03028234

[45] Glasser M L and Guttmann A J 1994 Lattice Green function (at 0) for the 4D hypercubic lattice *J. Phys. A* **27** 7011–7014 URL http://arxiv.org/abs/cond-mat/9408097

[46] Guttmann A J and Prellberg T 1993 Staircase polygons, elliptic integrals, Heun functions, and lattice Green functions *Physical Review E* **47** 2233–2236 URL http://www.maths.qmul.ac.uk/~tp/papers/pub004.pdf

[47] Joyce G S, Delves R T and Zucker I J 2003 Exact evaluation of the Green functions for the anisotropic face-centred and simple cubic lattices *J. Phys. A* **36** 8661–8672 URL http://dx.doi.org/10.1088/0305-4470/36/32/307

[48] Guttmann A J 2010 Lattice Green's functions in all dimensions *J. Phys. A* **43** 305205, 26 URL http://arxiv.org/abs/1004.1435

[49] Christol G 1990 Globally bounded solutions of differential equations *Analytic number theory (Tokyo, 1988)* (*Lecture Notes in Math.* vol 1434) (Berlin: Springer) pp 45–64 URL http://dx.doi.org/10.1007/BFb0097124

[50] Boukraa S, Hassani S and Maillard J-M 2012 Holonomic functions of several complex variables and singularities of anisotropic Ising n-fold integrals *J. Phys. A* **45** 33 pp. URL http://arxiv.org/abs/1207.1784

[51] Barouch E 1980 On the Ising model in the presence of magnetic field *Phys. D* **1** 333–337 URL http://dx.doi.org/10.1016/0167-2789(80)90031-7

[52] Guttmann A J 2009 Lattice Green functions and Calabi-Yau differential equations *J. Phys. A* **42** 232001, 6 URL http://dx.doi.org/10.1088/1751-8113/42/23/232001

[53] Arnold W I 2001 On teaching mathematics *Wiadom. Mat.* **37** 17–26 Translated from the Russian by Danuta Śledziewska-Błocka

[54] Arnold V I 1998 Sur l'éducation mathématique *Gaz. Math.* 19–29 URL http://smf4.emath.fr/Publications/Gazette/1998/78/smf_gazette_78_19-29.pdf

[55] Pochekutov D Y 2009 Diagonals of the Laurent series of rational functions *Sibirsk. Mat. Zh.* **50** 1370–1383 URL http://dx.doi.org/10.1007/s11202-009-0119-z

[56] Lipshitz L and van der Poorten A J 1990 Rational functions, diagonals, automata and arithmetic *Number theory (Banff, AB, 1988)* (Berlin: de Gruyter) pp 339–358

[57] Lipshitz L 1988 The diagonal of a D-finite power series is D-finite *J. Algebra* **113** 373–378 URL http://dx.doi.org/10.1016/0021-8693(88)90166-4

[58] Adamczewski B and Bell J P Diagonalization and Rationalization of algebraic Laurent series Preprint http://arxiv.org/pdf/1205.4090.pdf

[59] Christol G 1986-1987 Fonctions hypergéométriques bornées *Groupe de travail d'analyse ultramétrique* **14** 1–16 URL http://archive.numdam.org/article/GAU_1986-1987__14__A4_0.pdf

[60] Pólya G 1921–1922 Sur les séries entières, dont la somme est une fonction algébrique *Enseignement Math.* **22** 38–47 URL http://retro.seals.ch/digbib/view?rid=ensmat-001:1921-1922:22::220

[61] Cameron R H and Martin W T 1938 Analytic continuation of diagonals and Hadamard compositions of multiple power series *Trans. Amer. Math. Soc.* **44** 1–7 URL http://dx.doi.org/10.2307/1990100

[62] Hadamard J 1892 Essai sur l'étude des fonctions données par leur développement de Taylor *J. Math. Pures Appl.* **4** 101–186 URL http://portail.mathdoc.fr/JMPA/PDF/JMPA_1892_4_8_A4_0.pdf

[63] Flajolet P and Sedgewick R 2009 *Analytic combinatorics* (Cambridge: Cambridge University Press) ISBN 978-0-521-89806-5 URL http://algo.inria.fr/flajolet/Publications/AnaCombi/anacombi

[64] Deligne P 1984 Intégration sur un cycle évanescent *Invent. Math.* **76** 129–143 URL http://dx.doi.org/10.1007/BF01388496

[65] Bostan A, Chyzak F, van Hoeij M and Pech L 2011 Explicit formula for the generating series of diagonal 3D rook paths *Séminaire Lotharingien de Combinatoire* **B66a** URL http://www.emis.de/journals/SLC/wpapers/s66bochhope.html

[66] Christol G 1984 Diagonales de fractions rationnelles et équations différentielles *Study group on ultrametric analysis, 10th year: 1982/83, No. 2, Exp. No. 18* (Paris: Inst. Henri Poincaré) pp 1–10 URL http://archive.numdam.org/article/GAU_1982-1983__10_2_A4_0.pdf

[67] Christol G 1985 Diagonales de fractions rationnelles et équations de Picard-Fuchs *Study group on ultrametric analysis, 12th year, 1984/85, No. 1, Exp. No. 13* (Paris: Secrétariat Math.) pp 1–12 URL http://archive.numdam.org/article/GAU_1984-1985__12_1_A8_0.pdf

[68] Christol G 1988 Diagonales de fractions rationnelles *Séminaire de Théorie des Nombres, Paris 1986–87* (*Progr. Math.* vol 75) (Boston, MA: Birkhäuser Boston) pp 65–90

[69] Furstenberg H 1967 Algebraic functions over finite fields *J. Algebra* **7** 271–277 URL http://dx.doi.org/10.1016/0021-8693(67)90061-0

[70] Ihara Y 1974 Schwarzian equations *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **21** 97–118 URL http://repository.dl.itc.u-tokyo.ac.jp/dspace/bitstream/2261/6446/1/jfs210106.pdf

[71] Honda T 1981 Algebraic differential equations *Symposia Mathematica, Vol. XXIV (Sympos., INDAM, Rome, 1979)* (London: Academic Press) pp 169–204

[72] Koike M 1999 Hypergeometric polynomials over finite fields *Tohoku Math. J. (2)* **51** 75–79 URL http://dx.doi.org/10.2748/tmj/1178224854

[73] Deuring M 1941 Die Typen der Multiplikatorenringe elliptischer Funktionenkörper *Abh. Math.*

Sem. Hansischen Univ. **14** 197–272 URL http://dx.doi.org/10.1007/BF02940746

[74] Igusa J i 1958 Class number of a definite quaternion with prime discriminant *Proc. Nat. Acad. Sci. U.S.A.* **44** 312–314 URL http://dx.doi.org/10.1073/pnas.44.4.312

[75] Necer A 1997 Séries formelles et produit de Hadamard *J. Théor. Nombres Bordeaux* **9** 319–335 URL http://archive.numdam.org/article/JTNB_1997__9_2_319_0.pdf

[76] Fliess M 1974 Sur divers produits de séries formelles *Bull. Soc. Math. France* **102** 181–191 URL http://archive.numdam.org/article/BSMF_1974__102__181_0.pdf

[77] Hurwitz A 1898 Ueber die Entwickelungscoefficienten der lemniscatischen Functionen *Math. Ann.* **51** 196–226 URL http://dx.doi.org/10.1007/BF01453637

[78] Hurwitz A 1898 Sur un théorème de M. Hadamard *C. R. Math. Acad. Sci. Paris* **128** 350–353

[79] Lamperti J 1958 On the coefficients of reciprocal power series *Amer. Math. Monthly* **65** 90–94 URL http://www.jstor.org/stable/2308880

[80] Trjitzinsky W J 1930 On composition of singularities *Trans. Amer. Math. Soc.* **32** 196–215 URL http://dx.doi.org/10.2307/1989490

[81] Denef J and Lipshitz L 1987 Algebraic power series and diagonals *J. Number Theory* **26** 46–67 URL http://dx.doi.org/10.1016/0022-314X(87)90095-3

[82] Boukraa S, Hassani S, Maillard J-M and Zenine N 2007 Landau singularities and singularities of holonomic integrals of the Ising class *J. Phys. A* **40** 2583–2614 URL http://arxiv.org/abs/math-ph/0701016

[83] Sathaye A 1983 Generalized Newton-Puiseux expansion and Abhyankar-Moh semigroup theorem *Invent. Math.* **74** 149–157 URL http://dx.doi.org/10.1007/BF01388535

[84] Aparicio-Monforte A and Kauers M 2012 Formal Laurent series in several variables *Expo. Math.* To appear URL http://www.risc.jku.at/people/mkauers/publications/aparicio12.pdf

[85] Nickel B 1999 On the singularity structure of the 2D Ising model susceptibility *J. Phys. A* **32** 3889–3906 URL http://dx.doi.org/10.1088/0305-4470/32/21/303

[86] Nickel B 2000 Addendum to: "On the singularity structure of the 2D Ising model susceptibility" [J. Phys. A **32** (1999), 3889–3906] *J. Phys. A* **33** 1693–1711 URL http://dx.doi.org/10.1088/0305-4470/33/8/313

[87] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 Square lattice Ising model susceptibility: series expansion method and differential equation for $\chi^{(3)}$ *J. Phys. A* **38** 1875–1899 URL http://arxiv.org/abs/hep-th/0411051

[88] Almkvist G and Zudilin W 2006 Differential equations, mirror maps and zeta values *Mirror symmetry. V* (*AMS/IP Stud. Adv. Math.* vol 38) (Providence, RI: Amer. Math. Soc.) pp 481–515 URL http://arxiv.org/abs/math/0402386

[89] Morrison D R 1993 Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians *J. Amer. Math. Soc.* **6** 223–247 URL http://arxiv.org/abs/alg-geom/9202004

[90] Batyrev V V and van Straten D 1995 Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties *Comm. Math. Phys.* **168** 493–533 URL http://arxiv.org/abs/alg-geom/9307010

[91] Almkvist G, van Enckevort C, van Straten D and Zudilin W Tables of Calabi–Yau equations Preprint http://arxiv.org/pdf/math/0507430 and database http://enriques.mathematik.uni-mainz.de/CYequations/

[92] Chen Y H, Yang Y and Yui N 2008 Monodromy of Picard-Fuchs differential equations for Calabi-Yau threefolds *J. Reine Angew. Math.* **616** 167–203 With an appendix by Cord Erdenberger URL http://arxiv.org/abs/math/0605675

[93] André Y 2004 Sur la conjecture des p-courbures de Grothendieck-Katz et un problème de Dwork *Geometric aspects of Dwork theory. Vol. I, II* (Walter de Gruyter GmbH & Co. KG, Berlin) pp 55–112

[94] Almkvist G 2006 Calabi-Yau differential equations of degree 2 and 3 and Yifan Yang's pullback Preprint http://arxiv.org/abs/math/0612215

[95] Almkvist G 2007 Binomial identities related to Calabi–Yau differential equations Preprint http://arxiv.org/abs/math/0703255

[96] Zagier D 2009 Integral solutions of Apéry-like recurrence equations *Groups and symmetries* (*CRM Proc. Lecture Notes* vol 47) (Providence, RI: Amer. Math. Soc.) pp 349–366 URL http://people.mpim-bonn.mpg.de/zagier/files/tex/AperylikeRecEqs/fulltext.pdf

[97] Egorychev G P 1984 *Integral representation and the computation of combinatorial sums* (*Translations of Mathematical Monographs* vol 59) (Providence, RI: American Mathematical Society) ISBN 0-8218-4512-8 Translated from the Russian by H. H. McFadden, Translation edited by Lev J. Leifman

[98] Deligne P 1973 Deligne's letter to Piatetski-Shapiro URL http://www.math.ias.edu/~jaredw/DeligneLetterToPiatetskiShapiro.pdf

[99] Strehl V 1994 Binomial identities—combinatorial and algorithmic aspects *Discrete Math.* **136** 309–346 Trends in discrete mathematics URL http://dx.doi.org/10.1016/0012-365X(94)00118-3

[100] Schmidt A L 1995 Legendre transforms and Apéry's sequences *J. Austral. Math. Soc. Ser. A* **58** 358–375 URL http://dx.doi.org/10.1017/S1446788700038350

[101] Zudilin W 2004 On a combinatorial problem of Asmus Schmidt *Electron. J. Combin.* **11** Research Paper 22, 8 pp. (electronic) URL http://www.combinatorics.org/Volume_11/Abstracts/v11i1r22.html

[102] Kontsevich M and Zagier D 2001 Periods *Mathematics unlimited—2001 and beyond* (Berlin: Springer) pp 771–808 URL http://people.mpim-bonn.mpg.de/zagier/files/periods/fulltext.pdf

[103] Ayoub J 2012 Une version relative de la conjecture des périodes de Kontsevich-Zagier *Ann. of Math.* To appear URL http://www.math.uiuc.edu/K-theory/1010/rel-KZ.pdf

[104] Perron B 1989 Les cycles évanescents sont dénoués *Ann. Sci. École Norm. Sup. (4)* **22** 227–253 URL archive.numdam.org/article/ASENS_1989_4_22_2_227_0.pdf

[105] Hattori A and Kimura T 1974 On the Euler integral representations of hypergeometric functions in several variables *J. Math. Soc. Japan* **26** 1–16 URL http://projecteuclid.org/euclid.jmsj/1240435361

[106] Driver K A and Johnston S J 2006 An integral representation of some hypergeometric functions *Electron. Trans. Numer. Anal.* **25** 115–120

[107] Grothendieck A 1966 On the de Rham cohomology of algebraic varieties *Inst. Hautes Études Sci. Publ. Math.* 95–103

[108] Katz N M 1972 Algebraic solutions of differential equations (p-curvature and the Hodge filtration) *Invent. Math.* **18** 1–118 URL http://dx.doi.org/10.1007/BF01389714

[109] Nickel B, Jensen I, Boukraa S, Guttmann A J, Hassani S, Maillard J-M and Zenine N 2010 Square lattice Ising model $\tilde{\chi}^{(5)}$ ODE in exact arithmetic *J. Phys. A* **43** 195205, 24 URL http://arxiv.org/abs/1002.0161

[110] Dwork B, Gerotto G and Sullivan F J 1994 *An introduction to G-functions* (*Annals of Mathematics Studies* vol 133) (Princeton, NJ: Princeton University Press) ISBN 0-691-03681-0

[111] Beukers F and Heckman G 1989 Monodromy for the hypergeometric function ${}_nF_{n-1}$ *Invent. Math.* **95** 325–354 URL http://dx.doi.org/10.1007/BF01393900

[112] van Hoeij M and Weil J-A 2005 Solving second order linear differential equations with Klein's theorem *Proceedings of the 2005 international symposium on Symbolic and algebraic computation* ISSAC'05 (New York, NY, USA: ACM) pp 340–347 ISBN 1-59593-095-7 URL http://doi.acm.org/10.1145/1073884.1073931

[113] Bostan A, Boukraa S, Hassani S, Maillard J-M, Weil J-A, Zenine N and Abarenkova N 2010 Renormalization, isogenies, and rational symmetries of differential equations *Adv. Math. Phys.* Art. ID 941560, 44 URL http://arxiv.org/abs/0911.5466

[114] Atkin A O L and Morain F 1993 Elliptic curves and primality proving *Math. Comp.* **61** 29–68 URL http://dx.doi.org/10.2307/2152935

[115] Chan H H 1998 On Ramanujan's cubic transformation formula for ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ *Math. Proc. Cambridge Philos. Soc.* **124** 193–204 URL http://dx.doi.org/10.1017/S0305004198002643

[116] Maier R S 2005 On reducing the Heun equation to the hypergeometric equation *J. Differential Equations* **213** 171–203 URL http://arxiv.org/abs/math/0203264

[117] Golyshev V and Stienstra J 2007 Fuchsian equations of type DN *Commun. Number Theory Phys.* **1** 323–346 URL http://arxiv.org/abs/math/0701936

[118] Husemöller D 2004 *Elliptic curves* 2nd ed (*Graduate Texts in Mathematics* vol 111) (New York: Springer-Verlag) ISBN 0-387-95490-2 With appendices by Otto Forster, Ruth Lawrence and Stefan Theisen

[119] Bouw I I and Möller M 2010 Differential equations associated with nonarithmetic Fuchsian groups *J. Lond. Math. Soc. (2)* **81** 65–90 URL http://arxiv.org/abs/0710.5277

[120] Dettweiler M and Reiter S 2010 On globally nilpotent differential equations *J. Differential Equations* **248** 2736–2745 URL http://arxiv.org/abs/math/0605383

[121] Manin Y I 1999 *Frobenius manifolds, quantum cohomology, and moduli spaces* (*American Mathematical Society Colloquium Publications* vol 47) (Providence, RI: American Mathematical Society) ISBN 0-8218-1917-8

[122] Lian B H and Yau S T 1996 Arithmetic properties of mirror map and quantum coupling *Comm. Math. Phys.* **176** 163–191 URL http://arxiv.org/abs/hep-th/9411234

[123] Maier R S 2009 On rationally parametrized modular equations *J. Ramanujan Math. Soc.* **24** 1–73 URL http://arxiv.org/abs/math/0611041

[124] Maier R S 2007 The 192 solutions of the Heun equation *Math. Comp.* **76** 811–843 (electronic) URL http://arxiv.org/abs/math/0408317

[125] Cooper S 2009 Inversion formulas for elliptic functions *Proc. Lond. Math. Soc. (3)* **99** 461–483 URL http://dx.doi.org/10.1112/plms/pdp007

[126] Várilly-Alvarado A and Zywina D 2009 Arithmetic E_8 lattices with maximal Galois action *LMS J. Comput. Math.* **12** 144–165 URL http://arxiv.org/abs/0803.3063

[127] Rodriguez-Villegas F Integral ratios of factorials and algebraic hypergeometric functions Preprint http://www.math.utexas.edu/users/villegas/publications/oberwolfach-05.pdf (Summary of talk at Oberwolfach 2005)

[128] Chebyshev P L 1852 Mémoire sur les nombres premiers *J. de Math. Pures Appl.* **17** 366–390 URL http://archive.numdam.org/article/ASENS_1920_3_37__1_0.pdf

[129] Delaygue E 2011 *Propriétés arithmétiques des applications miroir* Ph.D. thesis Université de Grenoble URL http://tel.archives-ouvertes.fr/tel-00628016/

[130] Bober J W 2009 Factorial ratios, hypergeometric series, and a family of step functions *J. Lond. Math. Soc. (2)* **79** 422–444 URL http://arxiv.org/abs/0709.1977

[131] Katz N M 1990 *Exponential sums and differential equations* (*Annals of Mathematics Studies* vol 124) (Princeton, NJ: Princeton University Press) ISBN 0-691-08598-6; 0-691-08599-4

[132] McKean H and Moll V 1997 *Elliptic curves* (Cambridge: Cambridge University Press) ISBN 0-521-58228-8; 0-521-65817-9 Function theory, geometry, arithmetic

[133] Vidūnas R and Filipuk G 2009 Parametric transformations between the Heun and Gauss hypergeometric functions Preprint http://arxiv.org/abs/0910.3087

[134] Valent G 2007 Heun functions versus elliptic functions *Difference equations, special functions and orthogonal polynomials* (World Sci. Publ., Hackensack, NJ) pp 664–686 URL http://arxiv.org/abs/math-ph/0512006

[135] Treibich A and Verdier J-L 1990 Solitons elliptiques *The Grothendieck Festschrift, Vol. III* (*Progr. Math.* vol 88) (Boston, MA: Birkhäuser Boston) pp 437–480 With an appendix by J. Oesterlé URL http://dx.doi.org/10.1007/978-0-8176-4576-2_11

[136] Guttmann A J and Rogers M D 2012 Spanning tree generating functions and Mahler measures Preprint http://arxiv.org/abs/1207.2815

[137] Stanley R P 1999 *Enumerative combinatorics. Vol. 2* (*Cambridge Studies in Advanced Mathematics* vol 62) (Cambridge: Cambridge University Press) ISBN 0-521-56069-1; 0-521-78987-7 With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin

[138] Chyzak F 2000 An extension of Zeilberger's fast algorithm to general holonomic functions *Discrete Math.* **217** 115–134 URL http://dx.doi.org/10.1016/S0012-365X(99)00259-9

[139] Zeilberger D 1991 The method of creative telescoping *J. Symbolic Comput.* **11** 195–204 URL http://dx.doi.org/10.1016/S0747-7171(08)80044-2

[140] Koutschan C 2010 A fast approach to creative telescoping *Math. Comput. Sci.* **4** 259–266 URL http://arxiv.org/abs/1004.3314

[141] Koutschan C 2011 Lattice Green's Functions of the Higher-Dimensional Face-Centered Cubic Lattices Preprint http://arxiv.org/abs/1108.2164

[142] Schoutens H 1999 Embedded resolution of singularities in rigid analytic geometry *Ann. Fac. Sci. Toulouse Math. (6)* **8** 297–330 URL http://archive.numdam.org/article/AFST_1999_6_8_2_297_0.pdf

[143] Steenbrink J and Zucker S 1985 Variation of mixed Hodge structure. I *Invent. Math.* **80** 489–542 URL http://dx.doi.org/10.1007/BF01388729

[144] Golyshev V V 2007 Classification problems and mirror duality *Surveys in geometry and number theory: reports on contemporary Russian mathematics* (*London Math. Soc. Lecture Note Ser.* vol 338) (Cambridge: Cambridge Univ. Press) pp 88–121 URL http://arxiv.org/abs/math/0510287

[145] Atkin A O L 1992 The number of points on an elliptic curve (modulo a prime) E-mails to the URL https://listserv.nodak.edu/cgi-bin/wa.exe?A2=NMBRTHRY;d054c463.9202 and URL https://listserv.nodak.edu/cgi-bin/wa.exe?A2=NMBRTHRY;d6b18300.9207

[146] Morain F 1995 Calcul du nombre de points sur une courbe elliptique dans un corps fini: aspects algorithmiques *J. Théor. Nombres Bordeaux* **7** 255–282 Les Dix-huitièmes Journées Arithmétiques (Bordeaux, 1993) URL http://archive.numdam.org/article/JTNB_1995__7_1_255_0.pdf

[147] Elkies N D 1998 Elliptic and modular curves over finite fields and related computational issues *Computational perspectives on number theory (Chicago, IL, 1995)* (*AMS/IP Stud. Adv. Math.* vol 7) (Providence, RI: Amer. Math. Soc.) pp 21–76 URL http://www.math.harvard.edu/~elkies/modular.pdf

[148] Miller R L 2011 Proving the Birch and Swinnerton-Dyer conjecture for specific elliptic

curves of analytic rank zero and one *LMS J. Comput. Math.* **14** 327–350 URL http://arxiv.org/abs/1010.2431

[149] Kaneko M and Zagier D 1995 A generalized Jacobi theta function and quasi-modular forms *The moduli space of curves (Texel Island, 1994)* (*Progr. Math.* vol 129) (Boston, MA: Birkhäuser Boston) pp 165–172 URL http://www2.math.kyushu-u.ac.jp/~mkaneko/papers/genjacobi.pdf

[150] Cohn H 1988 Fricke's two-valued modular equations *Math. Comp.* **51** 787–807 URL http://dx.doi.org/10.2307/2008779

[151] Fricke R 1915 *Die elliptischen Funktionen und Ihre Anwendungen. Erster Teil.* (Leipzig: B. G. Teubner)

[152] Bertin M-J, Decomps-Guilloux A, Grandet-Hugot M, Pathiaux-Delefosse M and Schreiber J-P 1992 *Pisot and Salem numbers* (Basel: Birkhäuser Verlag) ISBN 3-7643-2648-4 With a preface by David W. Boyd URL http://dx.doi.org/10.1007/978-3-0348-8632-1

[153] Hibino T and Murabayashi N 1997 Modular equations of hyperelliptic $X_0(N)$ and an application *Acta Arith.* **82** 279–291 URL http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.113.3530&rep=rep1&type=pdf

[154] Almkvist G, van Straten D and Zudilin W 2011 Generalizations of Clausen's formula and algebraic transformations of Calabi-Yau differential equations *Proc. Edinb. Math. Soc. (2)* **54** 273–295 URL http://dx.doi.org/10.1017/S0013091509000959

[155] Vidūnas R 2005 Transformations of some Gauss hypergeometric functions *J. Comput. Appl. Math.* **178** 473–487 URL http://http://arxiv.org/abs/math/0310436

[156] Baxter R J 1980 Hard hexagons: exact solution *J. Phys. A* **13** L61–L70 URL http://stacks.iop.org/0305-4470/13/L61

[157] Joyce G S 1989 On the icosahedral equation and the locus of zeros for the grand partition function of the hard-hexagon model *J. Phys. A* **22** L237–L242 URL http://stacks.iop.org/0305-4470/22/L237

[158] Joyce G S 1988 On the hard-hexagon model and the theory of modular functions *Philos. Trans. Roy. Soc. London Ser. A* **325** 643–702 URL http://dx.doi.org/10.1098/rsta.1988.0077