Analytic combinatorics and combinatorial physics

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International Conference on Combinatorics and Physics
Karol A. Penson's 71'-Fest
Polish Academy of Sciences Scientific Center in Paris, 11-12 January 2018

Photo taken by Ira Gessel during the Séminaire Lotharingien of April 2005, for Xavier Viennot's 60th birthday.

Many links between physic and combinatorics seen these 2 days!

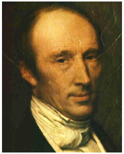
(Caveat: lists permuted & several matchings are possible!)

quantum physics	⋮
⋮	integer sequences (Bell, Catalan, Delannoy numbers...)
Schrödinger equation	non-crossing diagrams
chemical reactions	Toda Lattices
quarks	free probability
Hamiltonian	ordering
entanglement	shuffle product
Ising model	Heisenberg-Weyl algebra
nuclear reaction	cyclic groups
dissasociator	Pólya-Ehrenfest urn model
differential equations	combinatorial Hopf algebra
renormalization	fractional calculus
normal ordering	modular forms
universal laws	Latin squares
quantum optics	moment problems, orthogonal polynomials
crystals	number theory multiplicative functions
simulated annealing	domino tilings enumeration
transition phase	stable laws
⋮	⋮

Karol worked in an impressive large part [cf arXiv] of these topics, and also in analytic combinatorics!

Euler

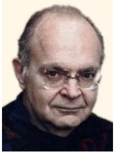
(1707-1783)



Cauchy

(1789-1857)

$\longrightarrow$



Knuth

(1938–)

Flajolet

(1948–2011)

• CombinatoricSSSSSSS (enumerative, bijective, algebraic, additive, topological, geometrical, extremal, additive, of words, . . .)

• Analytic combinatorics

Aim: Enumeration of finite/recursive structures, establish statistical behaviour.

Tools: generating functions and complex analysis.

discrete object (combinatorial structure counted by a_n)

$\Downarrow$

continuous (complex variable series $\sum_{n\geq 0} a_n z^n$)

$\Downarrow$

analysis ("Majorer, minorer, approcher", functional eq., closed forms)

asymptotics (singularities, saddle point, Mellin transform...)

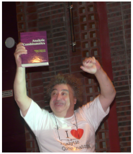
$\Downarrow$

properties of the discrete world (enumerative, limit laws...)

Applications: combinatorics, computer science, probability theory, number theory, biology (DNA), chemistry, statistical mechanics, . . .

The symbolic method

Schützenberger's credo (1920-1996): correspondence between combinatorial identities and functional identities. ($\approx$ non-commutative world to commutative world)



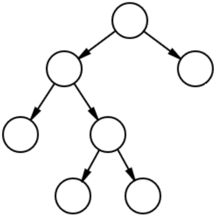
Flajolet et al.'s symbolic approach:
There exists a magic dictionary which translates
any combinatorial structure $\mathcal{A}$
into its generating function $A(z) = \sum_n a_n z^n$.

The magic dictionary:

product	$\mathcal{A} \times \mathcal{B}$	$\mapsto$	$A(z) \times B(z)$
sequence	$\mathrm{Seq}\, \mathcal{A}$	$\mapsto$	$\frac{1}{1-A(z)}$
set	$\mathrm{Set}\, \mathcal{A}$	$\mapsto$	$\exp(A(z))$
cycle	$\mathrm{Cyc}\, \mathcal{A}$	$\mapsto$	$\ln \frac{1}{1-A(z)}$
substitution	$\mathcal{B} \circ \mathcal{A}$	$\mapsto$	$B(A(z))$
inclusion − exclusion	$\mathcal{A}$(atom or nothing)	$\mapsto$	$A(z+1)$
pointing	$\Theta \mathcal{A}$	$\mapsto$	$z \frac{d}{dz} A(z)$

NB: This dictionary this explains Taylor formula, Lagrange inversion, . . . It also explains why combinatorial Hopf algebras lead to many nice explicit formulas.

Automatization: open source package `Combstruct` in Maple



b_n := number of binary trees with n (internal) nodes.

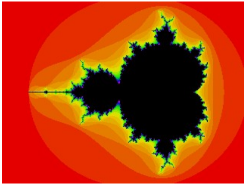
Counted by the generating function $B(z) = \sum_{n\geq 0} b_n z^n$

recursive definition leads to functional equation:

$$\mathcal{B} = \text{leaf} + \mathcal{B} \times \text{node} \times \mathcal{B} \Longrightarrow B(z) = 1 + zB^2(z)$$

$$B(z) = \frac{1-\sqrt{1-4z}}{2z} = 1 + z + 2z^2 + 5z^3 + 14z^4 + \ldots$$

One gets Catalan numbers: $\frac{\binom{2n}{n}}{n+1} \sim \frac{4^n}{\sqrt{\pi n^3}}$



Height of trees via Flajolet–Odlyzko's singularity analysis.

Mandelbrot iteration: $b^{[h+1]}(z) = z + b^{[h]}(z)^2$

H_n: cumulative height of trees of size n

$$H(z) = \sum_{n\geq 0} H_n z^n = \sum_{h\geq 0} B(z) - b^{[h]}(z)$$

$$H(z) = -2\ln(1-4z) + K + O\left(|1-4z|^{\nu}\right)$$

transfer theorem

$$\Rightarrow H_n \sim 2n^{-1}4^n \quad \Rightarrow H_n/B_n = 2\sqrt{\pi n} \quad \square$$

Universality: height of t-ary trees = $O(\sqrt{\pi n})$

(limit law $\approx$ Jacobi theta function).

permutations of $\{1,\ldots,n\}$ with k cycles

$$\mathcal{P} = Set(Cyc)$$

$$F(z,u) = \sum_{n\geq 0} f_n(u)\frac{z^n}{n!} = \exp\left(u \ln \frac{1}{1-z}\right)$$

$$\mu_n = \frac{\sum_{k\geq 0} k f_{n,k}}{f_n} = \frac{[\frac{z^n}{n!}]\partial_u F(z,u)_{|u=1}}{[\frac{z^n}{n!}]F(z,1)}$$

$$= \frac{[\frac{z^n}{n!}]\frac{1}{1-z}\ln(\frac{1}{1-z})}{n!} = \ln n + \gamma + \frac{1}{2n} + O(\frac{1}{n^2})$$

"A permutation has $\sim \ln n$ cycles in average"

$$\sigma_n = \sqrt{\ln n} + o(1)$$

$$\lim_{n\to\infty} \Pr\{X_n \leq \ln n + \gamma + x\sqrt{\ln n}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp^{-t^2/2}\, dt$$

$\rightsquigarrow$ Extension of Erdős–Kac theorem to many structures.

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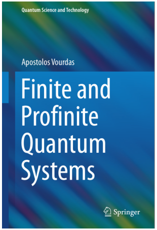
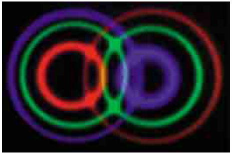
But before to present this, let me bribe the chairman to get more time :-)



Camille Jordan (1838-1922)

Theorem (Banderier–Vourdas):
Size of the symplectic group ($\det M = 1 \mod n$) :
$\#Sp(2, \mathbb{Z}_n) = nJ_2(n)$
proof via : $\#Sp(2, \mathbb{Z}_{p^e}) = p^{2e}\phi(p^e)(1 + 1/p)$



Quantum optics, anharmonic oscillator,
Weyl functions, Wigner functions... $\Longrightarrow$ Apostol Vourdas ;-)
Theorem (Banderier–Vourdas):
in order to realize this quantic tomography,
on average $\sim \dfrac{n^2}{3\zeta(3)}$ “lines” are enough.

Leonard Euler (1707-1783)

$f(n.m) = f(n).f(m)$ for n and m relatively primes

- $Id_k(n) = n^k$
- $\epsilon(1) = 1$ and $\epsilon(n) = 0$ for $n > 1$
- $\gcd(n, k)$

- Legendre symbol: $\left(\frac{n}{p}\right) = 1$ if n is a square $\bmod p$, -1 elsewhere (and 0 if $p|n$).
- Dirichlet characters $\chi_d(n) = \omega_n$ (where $\omega_n^{\phi(d)} = 1$)
- Euler totient function $\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^*$
- Möbius function $\mu(n) = \begin{cases} 1 & \text{if} & n = 1 \\ (-1)^r & \text{if} & n = p_1 \cdots p_r \\ 0 & \text{elsewhere} & \end{cases}$
- sum of divisors $\sigma_k(n) = \sum_{d|n} d^k$
- Liouville functions $\lambda(n) = a^{\Omega(n)}$ where $\Omega(n)$ is the number of prime factors of n (with or without their multiplicities)
- Dedekind ψ function $\psi(n) = n \prod_{p|n}(1 + 1/p)$
- Jordan function $J_k(n) = n^k \prod_{p|n}(1 - p^{-k})$, in particular: $J_1 = \phi$, $J_2(n) = \phi(n)\psi(n)$
- ...

Gustav Lejeune Dirichlet (1805-1859)

$$L(s,f)=\sum_{n\geq 1} f(n)n^{-s}$$

Convolution product:

$$L(s,f).L(s,g)=\sum_{n\geq 1}\left(\sum_{a.b=n} f(a)g(b)\right)n^{-s}=L(s,f*g)$$

Euler product formula:

$$L(s,f)=\prod_{p\in\mathcal{P}}\sum_{k=0}^{\infty}\frac{f(p^k)}{p^{ks}}=\prod_{p\in\mathcal{P}}\frac{1}{1-f(p)p^{-s}}$$

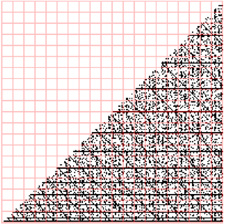
Some explicit formulae:

$$L(s,\epsilon)=1,\qquad L(s,1)=\sum_{n\geq 1}\frac{1}{n^s}=\zeta(s),\qquad L(s,Id_k)=\sum_{n\geq 1}\frac{n^k}{n^s}=\zeta(s-k)$$

$$L(s,\mu)=\frac{1}{\zeta(s)}$$ as $L(s,\epsilon)=L(s,\mu)L(s,1)$ (avatar of Möbius inversion formula)

$$L(s,\sigma_k)=L(s,Id^k)L(s,1)=\zeta(s-k)\zeta(s),\qquad L(s,J_k)=\frac{\zeta(s-k)}{\zeta(s)}$$

$$L(s,|\mu|)=\frac{\zeta(s)}{\zeta(2s)},\qquad L(s,\delta)=L(s,|\mu|)L(s,1)=\frac{\zeta(s)^2}{\zeta(2s)},\qquad L(s,\lambda)=\frac{\zeta(2s)}{\zeta(s)}$$



x is a residue k-th $\iff x \equiv y^k \mod n$.

$k = 2, 3, 4, 5 \cdots$: quadratic, cubic residues, quartic or biquadratic, quintic...

Example in $\mathbb{Z}_{10} = \mathbb{Z}/10\mathbb{Z}$:

x	1	2	3	4	5	6	7	8	9	10
x^2	1	4	9	6	5	6	9	4	1	0
x^3	1	8	7	4	5	6	3	2	9	0
x^4	1	6	1	6	5	6	1	6	1	0

the k-th residues in $\mathbb{Z}_{10}$ are thus:

- for $k = 2$: 0, 1, 4, 5, 6, 9
- for $k = 3$: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
- for $k = 4$: 0, 1, 5, 6

$\rho_k(n) :=$ number of k-th residues in $\mathbb{Z}/n\mathbb{Z}$.

$\rho_2(10) = 6, \rho_3(10) = 10, \rho_4(10) = 4$

Computational cost of $\rho_k(n)$: $\dfrac{n \ln(k)}{2 \ln 2}$ operations and a $O(n)$ amount of memory.

Is it possible to do better? Yes we can!

Local zeta functions, and links with combinatorics

Consider a system of polynomial equations,
their set of commun zeroes in a finite field $\mathbb{K}$ (with q^d elements) defines a variety V,
let N_d be this number of solutions.
The local zeta function is:

$$\zeta_{V,\mathbb{K}}(t) := \exp \sum_{d \geq 1} N_d \frac{t^d}{d}$$

It is therefore the Bell generating function in disguise.

$$\ln(\zeta_{V,\mathbb{K}}(t))' = \sum_{d \geq 1} N_d t^{d-1}$$

Link reminiscent of identities of Sparre Andersen, Spitzer for Brownian motion, Dvoretsky–Raney cycle lemma for Dyck paths:

$$S_n := X_1 + \cdots + X_n, \qquad \tau_n := \mathtt{Prob}\,[S_i \geq 0, i \in [\![1, n-1]\!], S_n < 0]$$

Erik Sparre Andersen's identity for ruin waiting time τ_n:

$$\sum_{n=1}^{\infty} \tau_n t^n = \exp\left(-\sum_{n=1}^{\infty} \frac{t^n}{n} \mathtt{Prob}\,[S_n < 0]\right)$$

$$\mathcal{B} = 1 + \mathcal{E}\Theta\mathcal{A} \qquad \Leftrightarrow \qquad B(z) = 1 + \frac{E'(z)}{E(z)}$$

André Weil
(1906-1998)

Bernard Dwork
(1923-1998)

Alexander Grothendieck
(1928-2014)

Pierre Deligne
(1944-)

In 1949, Weil conjectured:

- rationality of ζ_V, quotient of product of $(1 - ...)$ (Dwork, 1960)
- functional equation for ζ_V(Grothendieck, 1963)
- Riemann hypothesis for ζ_V (Deligne, 1973)
- Betti numbers related to ζ_V (Grothendieck, 1964)

Open question: what happens for rings ($\mathbb{Z}/q^n\mathbb{Z}$ instead of fields $\mathbb{F}_{q^n}$)?

Multiplicativity of ρ_k (the number of k-th powers modulo n)

Theorem

ρ_k is a multiplicative function: $\rho_k(nm) = \rho_k(n)\rho_k(m)$ (for n and m relatively prime).

Sketch of proof.

We construct a bijection between $(\mathbb{Z}_{nm})^k$ and $(\mathbb{Z}_n)^k \times (\mathbb{Z}_m)^k$.

$$Card((\mathbb{Z}_{nm})^k) = Card((\mathbb{Z}_n)^k).Card((\mathbb{Z}_m)^k)$$

$$\rho_k(nm) = \rho_k(n)\rho_k(m)$$

Injective, surjective... key tool = chinese remainder theorem:
For n and m relatively prime (so $un + vm = 1$ via Bézout's identity)

$$\left.\begin{array}{ll} x = a & \mod n \\ x = b & \mod m \end{array}\right\} \iff x = (aun + bvm) \mod nm$$

□

Theorem

$\rho_P(n) := \#\{x_i \in \mathbb{Z}_n : P(x_1, \ldots, x_k) = 0 \mod n\}$ is a multiplicative function.

(Caveat $\rho_{x_1^2+x_2^2-x_3}(n) \neq r_2(n) =$ nb of representations of n as sums of squares).

Theorem (recurrence, base 2: $m = 2^n$)

$$\rho_k(2^n) = \begin{cases} 1 & \textit{if} \quad n = 0 \\ 2 & \textit{if} \quad n = 1 \\ 1 + \dfrac{\phi(2^n)}{gcd(k,2)gcd(k,2^{n-2})} & \textit{if} \quad 2 \leq n < k \\ \rho_k(2^{n-k}) + \dfrac{\phi(2^n)}{gcd(k,2)gcd(k,2^{n-2})} & \textit{if} \quad n \geq k \end{cases}$$

Theorem (recurrence, base p: $m = p^n$)

for odd prime p,

$$\rho_k(p^n) = \begin{cases} 1 & \textit{if} \quad n = 0 \\ 1 + \dfrac{\phi(p^n)}{gcd(k,\phi(p^n))} & \textit{if} \quad 1 \leq n < k \\ \rho_k(p^{n-k}) + \dfrac{\phi(p^n)}{gcd(k,\phi(p^n))} & \textit{if} \quad n \geq k \end{cases}$$

Theorem (closed form, $k > 2$, odd p)

If $k < p$ and k is divisible by $p - 1$, one has:

$$\rho_k(p^n) = \begin{cases} 1 + \dfrac{(p-1)p^{n-1}(1-p^{-n})}{k(1-p^{-k})} & \text{if} \quad n = 0 \bmod k \\ 1 + \dfrac{(p-1)p^{n-1}\left(1-p^{-k([n/k]+1)}\right)}{k(1-p^{-k})} & \text{elsewhere} \end{cases}$$

Theorem (closed form, $k > 2$, odd p)

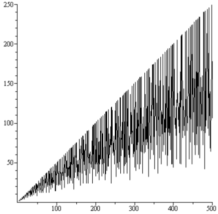
If $k < p$ and k is not divisible by $p - 1$, one has:

$$\rho_k(p^n) = \begin{cases} 1 + \dfrac{(p-1)p^{n-1}(1-p^{-n})}{1-p^{-k}} & \text{if} \quad n = 0 \bmod k \\ 1 + \dfrac{(p-1)p^{n-1}\left(1-p^{-k([n/k]+1)}\right)}{1-p^{-k}} & \text{elsewhere} \end{cases}$$

Theorem (closed form, $k > 2$, $p = 2$)

$$\rho_k(2^n) = \begin{cases} 1 + \dfrac{2^{n-1}(1-2^{-n})}{1-2^{-k}} & \text{if} \quad n = 0 \bmod k \\ 1 + \dfrac{2^{n-1}\left(1-2^{-k([n/k]+1)}\right)}{1-2^{-k}} & \text{elsewhere} \end{cases}$$

Ernesto Cesàro (1859-1906)



$\rho_2(n)$

If a_n has a chaotic behaviour,
one can consider $\sum_{k=1}^{n} a_k$.
In one sense, $\frac{\sum_{k=1}^{n} a_k}{n}$ will give
the "average order" of a_n.

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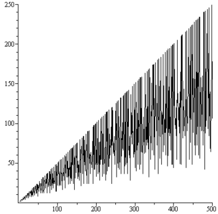

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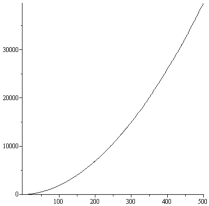

$\sum_{k=1}^{n} \rho_2(k)$

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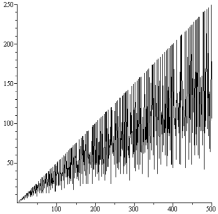
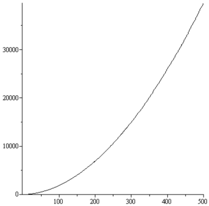
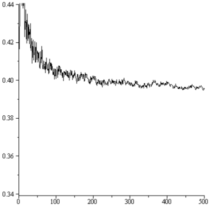


$\rho_2(n)$
very chaotic!



$\sum_{k=1}^{n} \rho_2(k)$
very smooth



$(\sum_{k=1}^{n} \rho_2(k))/(n^2/\sqrt{\ln(n)})$
converges to a constant?

Hubert Delange (1913-2003)
gave a way to get asymptotics
of Cesàro sums.

Theorem (tauberian theorem of Wiener–Ikehara–Delange (1930,31,54,63))

Let $F(s)$ be a Dirichlet series with coefficients $a_n > 0$ converging for $\Re(s) > \sigma > 0$. If $F(s) = \frac{A(s)}{(s-\sigma)^{\gamma+1}} + B(s)$ (with F analytic for $\Re(s) = \sigma, s \neq \sigma$), then

$$\sum_{n \leq N} a_n = \frac{A(\sigma)}{\sigma\Gamma(\gamma+1)} N^{\sigma} \ln^{\gamma} N \, (1 + o(1)).$$

NB: coherent with $\zeta(s)$.

Bernhard Riemann (1826-1866)

We have results for any k.
Enumeration for $k = 2$ [Stangl 1996].
We prove a conjecture of [Finch & Sebah 2006] on enumeration and asymptotics of cubes ($k = 3$).

$$F(s) = \zeta(s)\left(1 + \frac{2^{2s+1} - 2^{s+1} - 1}{2^{s+2}(2^{2s+1} - 2^{s-1} - 1)}\right)\prod_p \left(1 - \frac{(p^{s+1}+2)(p-1)}{2(p^{s+1}+1)(p^{s+1}-1)}\right) = G(s)\cdot\zeta(s)^{1/2}$$

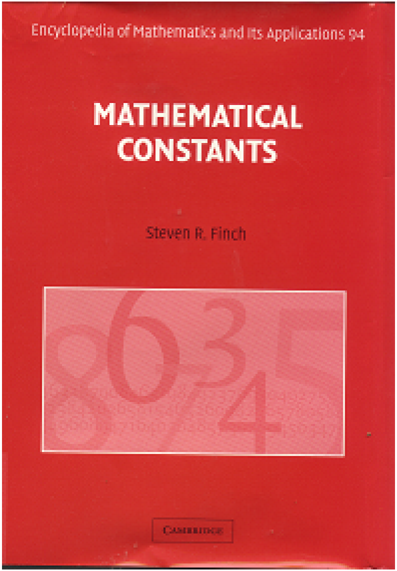
$$\sum_{n\le N} b(n) \sim C_2 \cdot N^2 \cdot (\ln N)^{-1/2} = (0.376...)N^2 \cdot (\ln N)^{-1/2}.$$

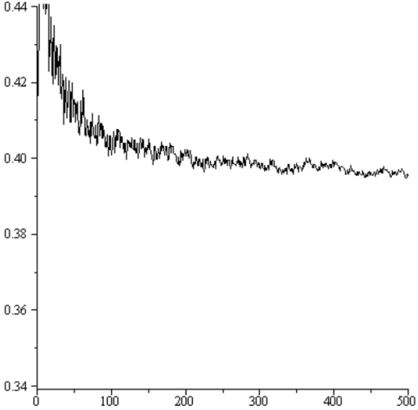
$$C_2 = \frac{17}{32}\frac{1}{\sqrt{\pi}}\prod_p \left(1 - \frac{p^2+2}{2(p^2+1)(p+1)}\right)\left(1-\frac{1}{p}\right)^{-1/2}$$

Theorem

$$\sum_{n\ge1} \frac{\rho_k(n)}{n^{s+1}} = G_k(s)\zeta(s)^{k/(k-1)} \qquad \sum_{n=1}^{N} \rho_k(n) \sim \frac{G_k(1)}{\Gamma((k-1)/k)}\frac{N^2}{2}(\ln N)^{-1/k}$$

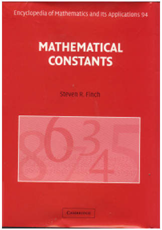
How to compute the value of the constant?

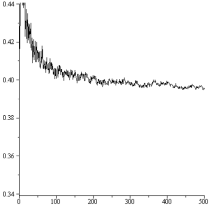




$\left(\sum_{k=1}^{n} \rho_2(k)\right) / \left(n^2 / \sqrt{\ln(n)}\right)$

How to compute the values of infinite products?





$$= \frac{\sum_{k=1}^{n} \rho_2(k)}{n^2/\sqrt{\ln(n)}}$$

$$\sum_{k\leq n} \rho_2(k) \sim C_2 \cdot n^2 \cdot (\ln n)^{-1/2}.$$

$$C_2 = \frac{17}{32}\frac{1}{\sqrt{\pi}}\prod_p \left(1 - \frac{p^2+2}{2(p^2+1)(p+1)}\right)\left(1-\frac{1}{p}\right)^{-1/2} \approx 0.376$$

Numerical scheme to compute these infinite products unpublished [Flajolet–Vardi, 96].

$$\sum_{n=1}^{\infty} f(\frac{1}{n}) = \sum_{m=2}^{\infty} f_m \zeta(m)$$

fast convergence e.g.: $\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = \frac{3}{2} - \ln 2 - \sum_{m=2}^{\infty}(-1)^m \frac{m-1}{m}(\zeta(m)-1)$

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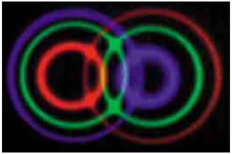
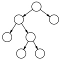
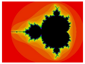
$\#Sp(2, \mathbb{Z}_n) = nJ_2(n)$
proof via : $\#Sp(2, \mathbb{Z}_{p^e}) = p^{2e}\phi(p^e)(1 + 1/p)$



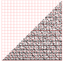
Quantum optics, anharmonic oscillator,
Weyl functions, Wigner functions... $\Longrightarrow$ Apostol Vourdas ;-)
Theorem (Banderier–Vourdas):
in order to realize this quantic tomography,
on average $\sim \dfrac{n^2}{3\zeta(3)}$ "lines" are enough.

[cf files .html, .mpeg]



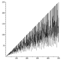
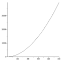
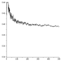



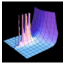


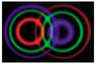







MATHEMATICAL
CONSTANTS


Conclusion

Karol, in conclusion, happy many more birthdays, articles, and music!