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Regularization or renormalization group: a few words to avoid confusion

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1 - Introduction

The motivation of the present talk is again to investigate the relations existing between the methods associated with the same name of "*renormalization*" but encountered in much different domains of theoretical physics. After having convinced myself of the close analogy that exists between renormalization analysis used in statistical mechanics and for chaotic dynamical systems, I tried to recognize their common principles and the associated technical steps in the chronologically first renormalization methods, conceived long before in quantum electrodynamics. I found it very difficult, until I understood that people speak of slightly different issues, hence of different associated tools; the name "renormalization" is in fact ambiguous, especially when used in the quantum field theory framework.

I will here try to disentangle the basic features of:

— renormalization understood in the sense of *regularization*, used mainly in quantum field theory for encompassing ultra-violet divergences,

— renormalization implemented as a *renormalization group*, used in various domains (including quantum field theory) to investigate critical phenomena, scaling behaviors and universal properties.

In both cases, one encounters *problems of divergences*, *singular perturbative expansions*, *ultra-violet cutoffs* and *effective parameters*; in both cases, the scales of the description play an important role. There is indeed conceptual and historical links between the two versions of renormalization, that I will now develop together with their main differences.

2 - Regularization and renormalization

2.1 - The two different "renormalizations"

The first meaning of the word "*renormalization*", coined in the middle of the century in quantum electrodynamics[**10**] (and even before in hydrodynamics), refers to a method, in fact rather a mathematical trick, in which some additional or external effects are taken into account through a *redefinition of the parameters of the model*. In this first sense, the adjective "*renormalized*" qualifies a coefficient or a quantity computed by adding to its actual value in the initial model for the system $\mathcal{S}$ some contributions of the same dimension, coming from additional phenomena taking place in $\mathcal{S}$, from interactions between $\mathcal{S}$ and its environment or from mechanisms not (or badly) taken into account in the model. A typical example is the case where some parameters of the initial theoretical model have no experimental reality; their values may be affected by the inadequacies of the model, mainly due to an unavoidable[1] lack of information on small scale mechanisms. These so-called "*bare*" parameters are replaced by effective ones in order to compensate, before doing calculations, the extra unphysical (sometimes divergent) terms of the model and to reproduce correctly the observed reality. The relevance of such a "re-normalization" is reinforced by the fact that the measured values correspond to these effective parameters.

The idea originates in the "renormalization" of the mass of a moving body into a moving fluid introduced in hydrodynamics in the last century. To illustrate this however limited meaning of the word "renormalization", I may also quote the "renormalization" of the mass of the classical electron. In classical electrodynamics, the electromagnetic mass m_0 is defined by $m_0c^2 = U$ where U is the total electromagnetic energy of the electron; m_0 diverges in the limit where the electron is described as a point-like charge.

[1] Constructing a model requires to choose a theoretical framework, hence to introduce minimum scales under which another theory has to be used; besides, such a truncation is necessary to obtain a workable model.

This kind of "renormalization" is then a formal operation for reestablishing the operational validity of a point-like model; it consists in compensating the divergence of m_0 (due to the model) by an infinite mass m' in order to obtain an effective finite mass $m_{eff} = m_0 - m'$, which actually gives the experimental value of the mass. A third example, that I will discuss with more details in the following section (§2.2), is the renormalization of the mass and charge of the electron used in quantum electrodynamics to get rid of apparent divergences. Instead of speaking of renormalized quantities, it seems better to me to speak of *effective* quantities. When it intends to encompass "theoretical" divergences or to handle seemingly diverging expansions, as in electrodynamics, the procedure is known as a "*regularization*".

The speeches you have already heard show that renormalization is often understood in a different and far more interesting meaning. This second meaning appeared in statistical mechanics, then extended to quantum field theory and, as I tried to show in my previous talk, to dynamical systems and various others domains of theoretical physics[**2**,**5**,**6**,**9**]. Here, renormalization is a powerful and constructive tool *devised to investigate scaling behaviors*; in fact, it achieves *to predict* explicit values for the critical exponents and *proves their universality* inside adequate classes of systems. Its major ingredient is a *transformation acting in the space of model systems*. The renormalization transformation is intended to be iterated and is designed so that its fixed points are "ideal" critical systems with exact scaling properties; its construction hence relies on the *conjectured scaling properties of the system*. It involves a coarse-graining of the small scale mechanisms and structures, but also rescalings in order to preserve[2] apparent scales, so that it makes sense to compare the system and the renormalized one.

The key-idea of renormalization is to replace the study of a given system by the study of the renormalization flow in the space of systems. It allows to split this space into universality classes with explicitely known features. I will show below that the set of renormalization transformations has a (semi)-group structure which allows to implement renormalization analysis in the *framework of group theory*[**4**,**8**].

2.2 - Ultra-violet divergences and ultra-violet cutoffs

Techniques for which the name "*renormalization*" was coined were introduced in quantum electrodynamics[**1**,**10**] to handle singular perturbative expansions. They have been renamed "*regularization*" after the extensive use of renormalization group and generalized renormalization methods. Such a regularization refers to a procedure intending to eliminate the so-called "*ultra-violet divergences*" arising in perturbative calculations with respect to interaction terms in the Hamiltonian[3], the zero order being the free-

[2]Such a conservation of characteristic scales and of physical invariants is necessary for having non-trivial fixed points.

[3]In quantum electrodynamics, the Hamiltonian is an Hermitian operator acting in a Hilbert space of wave functions, each representing a state (a configuration in statistical mechanics language) of the physical system.

particle Hamiltonian. To explain what are these ultra-violet divergences and where they come from, let me first recall that the adjective "*ultra-violet*" refers to high-energy and large-momentum domain in the conjugate space, in which structures and mechanisms are described that have short characteristic lengths and times.

The complex microscopic structures, up to sub-atomic scales, give rise to a lot of mechanisms, hence of interactions. Beyond the fact that it will be difficult, if not impossible, to describe all the possible coupling terms, it is not clear whether this explicit microscopic picture gives rise to finite contributions in the equations of evolution. This is the main origin of ultra-violet divergences. They show up the inadequacy of quantum electrodynamics when describing the high-energy processes, which is the consequence[4] partly of a lack of information on small space-time scales, partly of the simplifications and approximations which are necessary to obtain a workable theory. As a result, some quantities which are defined as approximate or limiting values in the theory may take infinite values, although their physical meaning indicates they have a finite value. The occurrence of ultra-violet divergences thus appears as a *theoretical* feature of quantum electrodynamics. Regularization intends to compensate the unphysical infinite contributions in perturbative expansions by adequately modifying the initial model (only its parameters if possible). Indeed, although it is impossible to handle a description in which each "virtual" microscopic contribution is taken into account separately, it should be possible to describe their resulting influence by means of effective contributions.

Let us note that the failure of "naive" perturbative approaches in quantum electrodynamics has not the same origin as in the case of a perturbative attempt to describe critical phenomena. In the latter case, failure occurs when carrying out two limiting behaviors $\beta \to \beta_c$ and $N \to \infty$ that cannot be inverted. In the present case, failure is due to the lack of information on the small scale physics which leads to ignore saturation or screening effects and makes impossible to distinguish between virtual mechanisms ans those that really occur.

One of the most efficient ways to implement regularization is to introduce an *ultra-violet cutoff* Λ thus ignoring large-momentum components $q \equiv ||\bar{q}|| > \Lambda$ for which the theory fails and produces unphysical divergences. In the real space, the corresponding procedure is a *coarse-graining* which bounds from below the length and time scales. Mechanisms involving energies (or momenta, up to a factor $1/c$) larger than Λ or exhibiting space-time structure of scales smaller than the minimum one are chosen *not to be described explicitely*, which is besides impossible within the framework of quantum electrodynamics if Λ is too high: only well known processes, for which quantum electrodynamics is valid, are mentioned. The processes left aside are taken into account through *low-energy contributions*, possibly infinite in order to counterbalance the divergences, if any. By construction, the effective model thus obtained has only components with $q \leq \Lambda$ and induces the same observable consequences as the initial model.

[4]Another difficulty stems from the fact that propagators are *generalized functions* (in the sense of Schwartz) so that their products are not defined, which causes many problems in the perturbative expansions.

Each transformation of a renormalization group also involves cutoffs and effective terms that reproduce the consequences of the modes left aside. However, it is implicitely assumed here that a cutoff $q \leq \Lambda$ has been performed as a preliminary step, that is, this renormalization group takes place *in a divergence-free theory*, with momenta and energies bounded above in order to reproduce the limitation of the experimentally accessible knowledge. Contrary to the cutoff involved in regularization, the cutoffs associated with the transformation of the group are *finite ones*, namely they amount to cutting out modes $\Lambda/k < q \leq \Lambda$ (for varying scale factor k).

2.3 - Renormalizability

In quantum electrodynamics language, a theory is said to be *renormalizable*[1] if its regularization (that is, adding effective terms to counterbalance apparent divergences due to the inadequacy of the theory at high energies) amounts only to a modification of the initial physical[5] parameters (charges, masses, coupling constants...) into "renormalized" or rather "effective" ones. Regularization is in that case a transformation defined in the parameter space. When regularization is implemented with an ultra-violet cutoff, renormalizability means that the overall influence of the modes left aside by the cutoff can be reproduced only in changing the parameters which control the equations for the remaining modes.

Renormalizability hence appears as a kind of internal consistency of the initial model, as its space-time scales can be bounded from below without increasing the dimension of the parameter space. It is still discussed[1] whether renormalizability has to be prescribed as a criterion of validity when constructing a quantum field theory.

If the theory is not renormalizable, introducing an ultra-violet cutoff leads not only to renormalizing initial parameters but also to adding new effective terms in order to reproduce faithfully the large scale consequences of the terms we intend to eliminate. Hence the minimum number of independent control parameters increase. This gives rise to difficulties when iterating the procedure, since an infinite-dimensional parameter space is ultimately involved (in particular when looking for fixed points).

2.4 - Emergence of the notion of renormalization group

The so-called *renormalization-group*[6] approach was first developed in quantum field theory. It was first introduced in 1953 by Stueckelberg and Petermann[7] to relate one to another the various effective models $\mathcal{S}_\Lambda$ obtained when canceling out the divergences of quantum electrodynamics using various ultra-violet cutoffs Λ. One may indeed in-

[5] This prescription about the "physical" character of the parameters prevents from formally including in the initial model additional terms having the same expression as the effective terms generated by the regularization but with parameters such that their resulting contribution equals zero. Indeed, in this manner, regularization amounts seemingly to modify parameters only, but in fact it also increases the number of parameters, that is, the dimension of the minimal parameter space. Regularization generates another model in that case, which invalidates the renormalizability of the initial model.

[6] Let us note that as most quantum field theories rely on group theory, it is most convenient that renormalization generates a group.

troduce a *transformation* $\mathcal{R}_k$ *relating the regularized models* $\mathcal{S}_\Lambda$ *and* $\mathcal{S}_{\Lambda/k}$. It follows from the constructive relation $\mathcal{R}_k[\mathcal{S}_\Lambda] = \mathcal{S}_{\Lambda/k}$ that the family $(\mathcal{R}_k)_{k\geq 1}$ satisfies the group-theoretic relation $\mathcal{R}_{k_1} \circ \mathcal{R}_{k_2} = \mathcal{R}_{k_1 k_2}$, which explain the term "renormalization group". To be precise, the operator relating $\mathcal{S}_\Lambda$ and $\mathcal{S}_{\Lambda/k}$ depends not only on k, but also on Λ: renormalization equation is better written: $\mathcal{S}_{\Lambda_2} = \mathcal{R}_{\Lambda_2,\Lambda_1}[\mathcal{S}_{\Lambda_1}]$. It follows that the group-theoretic structure writes $\mathcal{R}_{\Lambda_3,\Lambda_2} \circ \mathcal{R}_{\Lambda_2,\Lambda_1} = \mathcal{R}_{\Lambda_3,\Lambda_1}$: the renormalization group is *non-autonomous*.

Soon after, the same method was exploited fruitfully by Gell-Mann and Low[3] to study the behavior of quantum electrodynamics at small space-time scales. They first pointed out the relevance of using different *effective parameters according to the scale* of the description, which led them to introduce apparent charges $(e_\kappa)_\kappa$ for the electron, each value e_κ being specific to a model of maximum momentum scale κ. The breakthrough was to relate one to another the various values $(e_\kappa)_\kappa$ by a *renormalization transformation*. Introducing $E_Q = (e_{\kappa=\sqrt{Q}})^2$, renormalization acts on the charges according to:

$$R_{k,Q}[E_Q] = E_{kQ}$$

Group structure (non-commutative) shows up in the relation:

$$R_{k_2,k_1 Q} \circ R_{k_1,Q} = R_{k_1 k_2,Q}$$

Differentiating the relation $R_{k,Q}[E_Q] = E_{kQ}$ with respect to k in $k = 1$ gives the *(differential) renormalization equation* (m being the mass of the electron):

$$Q\ \left(\frac{dE_Q}{dQ}(Q)\right) = \left(\frac{dR_{k,Q}}{dk}\right)_{(k=1)}\ [E_Q] \equiv A(Q, E_Q) \equiv \psi\left(E_Q, \frac{m^2}{Q}\right)$$

The renormalization group is in that case associated with a nonlinear and non-autonomous dynamical system, thoroughly defined by:

$$\kappa^2\ \frac{de_\kappa^2}{d\kappa^2}\ =\ \psi\left(e_\kappa^2, \frac{m^2}{\kappa^2}\right)$$

A rather technical step is to compute the function ψ from the quantum electrodynamic description of the electron. This function ψ generates the renormalization group as it contains all the necessary information to write down the "trajectory under renormalization" of the charge, namely the correspondence $\kappa \mapsto e_\kappa$. Two limiting cases appeared at the ends of the sequence $(e_\kappa)_\kappa$:

— the bare charge e_∞, that appears in the initial theoretical description as this one includes arbitrarily high momenta,

— the experimental charge e_0, that can be measured by a macroscopic observer.

3 - Renormalization group

Let me now present a few technical details about the renormalization analysis supported by the group-theoretic structure of the set of renormalization operators[5,9] (in the second meaning of the word "renormalization").

3.1 - Group structure of the renormalization transformations

Let $\mathcal{S}$ be a physical system, or rather a *model* describing a real system in a *given theoretical framework*. In fact, $\mathcal{S}$ refers to a family $(\mathcal{S}_{\bar{N}})_{\bar{N}}$ of systems which differ by the values of a complete set $\bar{N} = (N_1, \ldots, N_m)$ of m *independent extensive variables*, for example numbers of particles, sizes, durations or maximal amplitudes. Renormalization $\mathcal{R}$ of $\mathcal{S}$ includes *rescalings in real and phase spaces*. Hence, it reduces each variable N_j by a factor $k_j > 0$, which defines the scale factors $\bar{k} = (k_1, \ldots, k_m)$ labelling each possible renormalization transformation $\mathcal{R}_{\bar{k}}$ of $\mathcal{S}$. *Internal consistency* requires that all the renormalization procedures associated with the same rescalings (that is, with the same value of $\bar{k}$) coincide, which can be expressed in the group-theoretic relation:

$$\mathcal{R}_{\bar{k}_1} \circ \mathcal{R}_{\bar{k}_2} = \mathcal{R}_{\bar{k}_1 \times \bar{k}_2} \qquad (\times \text{ being performed pairwise})$$

The set $[\mathcal{R}_{\bar{k}}]_{\bar{k}}$ of renormalization transformations of the system $\mathcal{S}$ is thus endowed with a *multiplicative*[7] *group-theoretical structure* (or at least a semi-group structure in case of a constraint $k_j \geq 1$). This group is known as the "*renormalization group*" associated with the renormalization procedure considered; in case of a continuous parametrization $\bar{k} \to \mathcal{R}_{\bar{k}}$, it is homomorphic to $(]0, \infty[^m, \times)$ (the product $\times$ being performed pairwise). This homomorphism $\bar{k} \to \mathcal{R}_{\bar{k}}$ ensures that the renormalization group is *commutative*. Iterating $\mathcal{R}_{\bar{k}}$ amounts to take successive powers $[\bar{k}]^{\times n}$ as scales factors labelling $\mathcal{R}$. In particular, for $m = 1$, describing the limiting behavior $\lim_{n\to\infty} \mathcal{R}^n_{k_0}$ for arbitrary $k_0 > 1$ can be achieved in studying $\lim_{k\to\infty} \mathcal{R}_k$ (weak convergence). If only one renormalization transformation $\mathcal{R}$ is constructed, the renormalization group is the trivial semi-group $\{\mathcal{R}^n, n \in \mathbf{N}\}$ generated by $\mathcal{R}$.

Concretely, the renormalization $\mathcal{R}_{\bar{k}}$ of a physical system $\mathcal{S}$ has to be formulated explicitely through *renormalization-group equations*, which express the action of a *renormalization operator* $R_{\bar{k}}$ on a *structure rule*[8] ϕ containing all the information required to describe the microscopic dynamics. The basic relation which *defines* $R_{\bar{k}}$ is:

$$R_{\bar{k}}[\phi(\mathcal{S})] = \phi[\mathcal{R}_{\bar{k}}(\mathcal{S})]$$

where each side has to be expressed analytically. It follows that $(R_{\bar{k}})_{\bar{k}}$ is also a multiplicative group, homomorphic to the underlying renormalization group $(\mathcal{R}_{\bar{k}})_{\bar{k}}$.

[7] Changes $(K_j = \ln k_j)_{j=1\ldots m}$ make the renormalization group homomorphic to the additive group $(\mathbf{R}^m, +)$.

[8] The *structure rule* ϕ of $\mathcal{S}$ is a "microscopic" state function (defined on the microscopic phase space of $\mathcal{S}$) which thoroughly determines the microscopic behavior of the system. For example, ϕ is the Hamiltonian function in classical statistical mechanics, the in a quantum field theory, the evolution map for a discrete dynamical system or the vector-field for a continuous dynamical system. The set $\Phi = \{\phi\}$ can be viewed as the "*space of systems*".

Renormalization-group equations are functional equations as soon as the structure rules ϕ belong to a functional space Φ. The property for $\phi^* \in \Phi$ of being a fixed point expresses a *functional self-similarity* of ϕ^*; such a renormalization is sometimes called a "functional self-similarity transformation". The simplest example is the renormalization used for a discrete dynamical system of evolution map f:

$$\Phi = \{f : \mathcal{X} \to \mathcal{X}\} \qquad\qquad Rf(x) = \frac{1}{\lambda}\, f \circ f(\lambda x) \qquad (|\lambda| < 1)$$

In this example, the renormalization group is the discrete semi-group generated by R. For a continuous flow $(f_t)_t$ generated by a vector field $V(.)$ through the differential equation $\partial_t[f_t(x_0)] = V[f_t(x_0)]$ with initial condition $f_0(x_0) = x_0$, typical renormalization writes:

$$[R_k f]_t(x) \;=\; \frac{1}{\lambda_k}\, f_{kt}(\lambda_k x) \qquad \text{with} \qquad \lambda_{k_1}\lambda_{k_2} = \lambda_{k_1 k_2}$$

It is more relevant to express the renormalization action on the structure rule, which is the vector field V in the present case. As the renormalized vector field $\mathcal{R}_k V$ must be the vector field generating the renormalized flow $(R_k f_t)_{t\geq 0}$, it is obvious that:

$$(\mathcal{R}_k V)(z) = (k/\lambda_k)\, V(\lambda_k z)$$

The renormalization action $\mathcal{R}_{\bar{k}}$ on any state function X typically expresses in the form:

$$[\mathcal{R}_{\bar{k}}(X)]_{\bar{N}} = R^X_{\bar{k}}[X_{\bar{k}\times\bar{N}}]$$

It defines the *renormalized state function* $\mathcal{R}_{\bar{k}}(X)$ if the *transformation* $R^X_{\bar{k}}$ *of the values of* X can be expressed analytically according to the different "concrete" steps of the renormalization procedure (coarse-grainings or cutoffs, averages, rescalings...). The label X indicates that the expression of $R^X_{\bar{k}}$ is specific to the state function X. Hence again, internal consistency requires that:

$$R^X_{\bar{k}_1} \circ R^X_{\bar{k}_2} = R^X_{\bar{k}_1\times\bar{k}_2}$$

When such a quantity I is associated with a *physical invariant* $\mathcal{I}$ (nevertheless depending on extensive variables $\bar{N}$), it must be preserved under renormalization action; it means that the consequence on I of the renormalization of the system is merely a rescaling of the extensive variables $\bar{N}$ which writes, for any possible value of $\bar{k}$:

$$\mathcal{R}_{\bar{k}}(I) = I \quad\Longleftrightarrow\quad I_{\bar{N}} = R^I_{\bar{k}}[I_{\bar{k}\times\bar{N}}]$$

Taking $k_j = 1/N_j$ $(j = 1 \ldots m)$ leads to an explicit scaling law for $I(\bar{N})$. If I is moreover a scale-independent quantity, $I_{\bar{N}}$ is independent of $\bar{N}$, hence physical invariance implies that $R^I_{\bar{k}}(I) = I$.

To summarize, renormalization corresponds to the physical transformation:

$$N_j \longrightarrow N_j/k_j \;\; (j = 1 \ldots m) \qquad \phi \longrightarrow R_{\bar{k}}\phi \qquad X \longrightarrow R^X_{\bar{k}}(X)$$

where as before, $N_1, \ldots N_m$ are extensive variables, ϕ a structure rule and X a state function. For a state function $\mathcal{X}$ that can be deduced from the knowledge of ϕ and of extensive variables $\bar{N}$ as a function $\mathcal{X} = X(\phi, \bar{N})$, the "secondary" renormalization operator $R^X_{\bar{k}}$ is prescribed by the knowledge of $R_{\bar{k}}$. Indeed, the definition $\mathcal{X}(\mathcal{S}_{\bar{N}}) = X(\phi, \bar{N})$ and the constructive relation $\mathcal{R}_{\bar{k}}[\mathcal{X}(\mathcal{S})] \equiv \mathcal{X}[\mathcal{R}_{\bar{k}}(\mathcal{S})]$ lead to:

$$R^X_{\bar{k}}[X(\phi, \bar{k} \times \bar{N})] \;\equiv\; X(R_{\bar{k}}\phi, \bar{N})$$

which reduces to $I(\phi) = I(R_{\bar{k}}\phi)$ for a physical invariant $\mathcal{I}$. The following step of the renormalization analysis is to look for *fixed points of the whole set of transformations of the group*. In case of a discrete group $\{R^n, n \geq 0\}$, the fixed points are merely fixed points of the "map" R in Φ: $R\phi^* = \phi^*$. In case of a group $(R_{\bar{k}})_{\bar{k}}$, ϕ^* must be a fixed point of any[9] operator of the group: $\forall \bar{k}, R_{\bar{k}}\phi^* = \phi^*$. Such fixed points correspond to systems exhibiting *exactly self-similar properties*, which express in an *exact scaling behavior* of its observable state functions: $R_{\bar{k}}\phi^* = \phi^*$ for any $\bar{k}$ hence $\mathcal{R}_{\bar{k}}\mathcal{S}^*$ is identical to $\mathcal{S}^*$ so that $\mathcal{R}_{\bar{k}}(X^*) = X^*$ (where $X^* \equiv X(\phi^*)$), which analytically expresses as follows:

$$\forall \bar{k} \qquad R^X_{\bar{k}}[X^*(\bar{k} \times \bar{N})] \;=\; X^*(\bar{N})$$

For example, if $m = 1$ and $R^X_k = k^{\alpha_X}\, X$, it leads to:

$$k^{\alpha_X}\, X^*(kN) = X^*(N) \quad \Longleftrightarrow \quad X^*(N) = N^{-\alpha_X}\, X^*(1)$$

3.2 - Infinitesimal renormalization

Pointing out the group structure of the renormalization group is of no benefit unless it is a *continuous group endowed with a differentiable structure*, known as a *Lie group*. Let us first consider the case of a one-dimensional Lie (semi)-group $(R_k)_{k>0}$. This group and its action can be thoroughly described in terms of the properties of the *infinitesimal generator*:

$$A \equiv \frac{dR_k}{dk}(k = 1)$$

acting on the space Φ of structure rules. In case of linear operators, R_k is given by $R_k = \exp[(\ln k)A] = \sum_{n=0}^{\infty} (\ln k)^n/n!\; A^n$. Let us note that R_k and A are either *both nonlinear operators*, either *both linear operators*, acting in the same space Φ. The fixed-point equations $[\forall k > 0, R_k\phi^* = \phi^*]$ equivalently express in a compact way: $A\phi^* = 0$. Moreover, *linear analysis around ϕ^* can be performed on A*, as:

$$[\forall k > 0, \quad DR_k(\phi^*)(\varphi) = k^\gamma\, \varphi\,] \quad \Longleftrightarrow \quad DA(\phi^*)(\varphi) = \gamma\, \varphi$$

[9] Otherwise the relevant renormalization group is the set of the operators $R_{\bar{k}}$ admitting ϕ^* as a fixed point (which is obviously a sub-group).

where the eigenvector φ of $DA(\phi^*)$ belongs to the vector space tangent to Φ in ϕ^*. Let us stress that due to the group structure, all the operators $[DR_k(\phi^*)]_{k>0}$ commute, hence have identical eigenvectors; moreover, any eigenvalue of $DR_k(\phi^*)$ is necessarily of the form k^γ, (the exponent γ being independent of k). These results show that infinitesimal analysis, whenever possible, is sufficient to describe the limiting behavior under renormalization action, in particular for determining (irr)relevant directions. For example, the stable directions of R_k, which describe the irrelevant perturbations, are given by the eigenvectors of $DA(\phi^*)$ associated with negative eigenvalues $\gamma_j < 0$.

Let us note that *renormalization defines a dynamical system in* Φ. This one is discrete for an isolated operator R which appears as its evolution map; it generates a trajectory $(\phi_n)_{n\geq 0}$ in Φ from the initial structure rule ϕ_0 according to $\phi_{n+1} = R\phi_n = R^{n+1}\phi_0$.

The associated dynamical system is continuous in case of a Lie renormalization group (or subgroup) of dimension 1; its infinitesimal operator A is then related to the vector field of the renormalization flow after a change $K = \ln k$ of parameter:

$$\widetilde{R}_K \equiv R_{e^K} \qquad \Longrightarrow \quad \text{additive group}, \quad A = \frac{d\widetilde{R}_K}{dK}(K=0)$$

$$\phi_K \equiv \widetilde{R}_K(\phi_0) = \widetilde{R}_{K-K_0}(\phi_{K_0}) \qquad \Longrightarrow \qquad \left(\frac{d\phi_K}{dK}\right)(K_0) = A[\phi_{K_0}]$$

Thus, a one-dimensional renormalization group is a continuous dynamical system where the time is replaced by the logarithm of the scale factor; stationarity or periodicity of a solution then have the meaning of "self-similarity".

Turning to state functions, the relation $R_k^X[X(N)] = X(N/k)$ stated above for a physical invariant X leads to the following differential formulation, known as a *renormalization equation*:

$$-N\,\frac{dX_N}{dN} = A^X(X_N) \qquad \text{where} \qquad A^X \equiv \frac{dR_k^X}{dk}(k=1)$$

If the physical quantity X is moreover scale-independent, the equation simplifies into the infinitesimal fixed-point equation $A^X(X^*) = 0$ (of course equivalent to the fixed-point equations $[\forall k > 0,\ R_k^X[X^*] = X^*]$).

In some cases, the renormalization operator still depends on the scale Q at which is acts; for example:

$$R_{k,Q}\,(\phi_Q) = \phi_{kQ}$$

Group structure shows up in the relation:

$$R_{k_2,k_1Q} \circ R_{k_1,Q} = R_{k_2k_1,Q}$$

As a consequence, commutativity fails[10] since $R_{k_2,k_1Q} \circ R_{k_1,Q} = R_{k_1,k_2Q} \circ R_{k_2,Q}$. Such a renormalization group is said to be *non-autonomous* as the associated infinitesimal renormalization equation appears to be a non-autonomous dynamical system. Differentiating the relation $R_{k,Q}(\phi_Q) = \phi_{kQ}$ with respect to k in $k = 1$ leads to the infinitesimal formulation:

$$Q\ \left(\frac{d\phi_Q}{dQ}\right)(Q) = \left(\frac{dR_{k,Q}}{dk}\right)_{(k=1)}\ [\phi_Q] \equiv A(Q, \phi_Q)$$

It unveils the non-autonomous character of the renormalization group, as the infinitesimal operator A still depends on the scale variable Q (standard case corresponds to the autonomous situation where $R_{k,Q}$ does not depend on Q). Let us note that in this non-autonomous case, the relevant eigenvalue equation writes:

$$DR_{k,Q}(\phi^*)[\varphi_Q] = \lambda_{k,Q}\varphi_{kQ}$$

which generates one-parameter families of "generalized eigenvectors" $[\varphi_Q^{(j)}]_Q$ ("generalized" in the sense that they still depend on a variable Q which is transformed in kQ under the action of $DR_{k,Q}$). The group structure implies that the associated two-parameter families of "generalized eigenvalues" $[\lambda_{k,Q}^{(j)}]_{k,Q}$ satisfy:

$$\lambda_{k_2,k_1Q}^{(j)} \times \lambda_{k_1,Q}^{(j)} = \lambda_{k_1k_2,Q}^{(j)}$$

The formulae expressed above in the case where only one independent extensive variable N (hence only one scale factor k) arises in $\mathcal{R}$, extend to any finite number m of independent variables $\bar{N} = (N_1, \ldots, N_m)$, which involves an m-parameter renormalization group $(\mathcal{R}_{\bar{k}})_{\bar{k}}$, homomorphic to $(]0, \infty[^m, \times)$ (the product $\times$ being performed pairwise). If it is a Lie group that is, if it is endowed with a differentiable structure, the associated Lie algebra is a m-dimensional[11] vector space of operators on Φ generated by m infinitesimal renormalization operators $A_j = (\partial R_{\bar{k}}/\partial k_j)(\bar{k}_0)$ where $\bar{k}_0 = (1 \ldots 1)$. It is to note that $R_{\bar{k}} = \prod_{j=1}^m r_{k_j}$ where $r_{k_j} = R_{1,\ldots,k_j,\ldots 1}$. These operators $(r_{k_j})_{j=1\ldots m}$ commute and hence $A_j = (dr_{k_j}/dk_j)(k_j = 1)$. The m operators $(A_j)_{j=1\ldots m}$ commute hence have the same eigenvectors, which also coincide with the eigenvectors of any $R_{\bar{k}}$ (being thus $\bar{k}$-independent). One obtains relevant *subgroups*, say of dimension n, when relating the m scale factors to only n independent ones. For example, one defines a one-dimensional subgroup by taking $k_j = k^{\alpha_j}$ and varying k in $]0, \infty[$; the fixed exponents $(\alpha_j)_{j=1\ldots m}$ characterizes this subgroup. The associated Lie algebra is the one-dimensional vector space generated by $A(\bar{\alpha}) = \sum_{j=1}^m \alpha_j A_j$.

[10] The situation is well-suited in quantum field theory, where non-commutativity of some pairs of operators is one of the conceptual basis of quantum mechanics.

[11] In that case, one says that the renormalization group has *dimension* m, which means that m is the minimal number of independent parameters necessary for labelling the renormalization operators through a one-to-one and homomorphic correspondence.

3.3 - Renormalization-group symmetry

Let $\mathcal{S}_0$ be a physical system depending on some parameter(s) e_0. A renormalization procedure is deviced, involving for example an ultra-violet cutoff; as explained in §3.1, a subset c of independent scale factors is chosen to label the renormalization operator R_c, which acts on $\mathcal{S}_0$ through a transformation:

$$\mathcal{S}_0(e_0) \mapsto \mathcal{R}_c[\mathcal{S}_0(e_0)] \equiv \mathcal{S}(c, e_0)$$

hence the semi-group relation reads

$$\mathcal{R}_{c_2}[\mathcal{S}(c_1, e_0)] = \mathcal{S}(c_2 \times c_1, e_0)$$

The *renormalization-group symmetry*[12] reads:

$$\mathcal{S}(c, e_0) = \mathcal{S}_0[e(c, e_0)] \qquad \text{hence} \qquad e[c_2, e(c_1, e_0)] = e(c_2 \times c_1, e_0)$$

with $\mathcal{S}(c_0, e_0) = \mathcal{S}_0(e_0)$ and $e(c_0, e_0) = e_0$ for a reference value c_0 associated with an infinite cutoff and scale factors equal to 1. In case of a state function $X_0(e_0)$, in particular the structure rule ϕ_0, renormalization-group symmetry implies that

$$R_c^X[X_0(e_0)] \equiv X(c, e_0) = X_0[e(c, e_0)]$$

Renormalization-group equations are straightforwardly deduced by differentiating the relation with respect to c, provided $(R_c^X)_c$ is a Lie (semi)-group:

$$\left(\frac{dR_c^X}{dc}\right)(c)[X_0(e_0)] = \left(\frac{\partial X}{\partial c}\right)(c, e_0) = \left(\frac{dX_0}{de}\right)[e(c, e_0)] \,.\, \left(\frac{\partial e}{\partial c}\right)(c, e_0)$$

Renormalization-group symmetry means that it is possible to transfer the renormalization of the system $\mathcal{S}_0$ (hence of all the state functions X) on a transformation of some parameters of $\mathcal{S}_0$: it is the mathematical formulation of renormalizability (§2.3), where the notion is extended to renormalization groups.

If c has only one component:

$$\left(\frac{dR_c^X}{dc}\right)(c) = \frac{1}{c}\, A^X \circ R_c^X \qquad \text{with} \qquad A^X = \left(\frac{dR_c^X}{dc}\right)(c_0)$$

For c having m independent components $(c_j)_{1 \le j \le m}$, the derivative d/dc generates a m-vector; the Lie group $(R_c^X)_c$ has dimension m, as it possesses m infinitesimal generators $(A_j^X)_{1 \le j \le m}$. For $j = 1 \ldots m$ and $c = c_0$:

$$A_j^X[X_0(e_0)] \equiv \left(\frac{\partial X}{\partial c_j}\right)(c_0, e_0) = \left(\frac{dX_0}{de}\right)(e_0) \,.\, \left(\frac{\partial e}{\partial c_j}\right)(c_0, e_0) \quad \text{with} \quad A_j^X = \left(\frac{\partial R_c^X}{\partial c_j}\right)(c_0)$$

[12] By construction, the renormalization operator dependence on the parameters c satisfies the *group-theoretic relation* $\mathcal{R}_{c_2} \circ \mathcal{R}_{c_1} = \mathcal{R}_{c_2 \times c_1}$ or more generally, in case of a non-autonomous renormalization equation: $\mathcal{R}_{c_2, e_1} \circ \mathcal{R}_{c_1, e_0} = \mathcal{R}_{c_2 \times c_1, e_0}$ with $e_1 = e(c_1, e_0)$.

It generalizes the renormalization-group equation obtained in the previous section §3.2 for any physical invariant X in the special case where $e_0 = \bar{N}$, $c = \bar{K}$ and $e(k, N) = [N_j/k_j]_{j=1...m}$. Renormalization-group symmetry appears as a property of invariance with respect to an adequate transformation of state variables, state functions and parameters:

$$\mathcal{S} \to \mathcal{R}_c(\mathcal{S}) \qquad e_0 \to e(c, e_0) \qquad X \to R_c^X(X)$$

such that the associated symmetry group is precisely the renormalization group. It reveals some scale invariance or more generally some self-similarity of the system.

I have just sketched here the notion of renormalization group and its basic applications, without speaking of its *representations*. Roughly, a representation describes a set of observable consequences of the group action. If it is irreducible, it can be identified with exactly one physical property. We have in fact encountered representations of the renormalization group above, as the relation $\mathcal{R}_k[X(\phi)] = X(R_k\phi)$ defines a representation $R_k \mapsto \mathcal{R}_k$ of the renormalization group in the space of state functions. It is decomposable, since each set of state functions X associated with a given physical quantity $\mathcal{X}$ (that is, having the same physical meaning or equivalently the same physical dimension) is globally invariant; the restriction of $\mathcal{R}_k$ to such a space defines another representation $R_k \mapsto R_k^X$ of the renormalization group.

As renormalization preserves all the symmetries of the physical system, *renormalization operators commute with any transformation of a symmetry group* of the system[13]. Renormalization action and symmetries are thus described in the same formalism of group theory. As representations have the same commutativity properties than the underlying group, the operators belonging to some representation of the renormalization group and the operators belonging to a representation of a symmetry group defined on the same vector space will have the same eigenvectors. The most important result is the *correspondance between universality classes and symmetry groups*. Indeed, systems belonging to the same universality class are related to the (same) typical representative of the class through renormalization, which preserves the symmetries; in consequence, they exhibit the same symmetry properties.

[13] If $(S_a)_{a>0}$ is a one-parameter symmetry group and ϕ^* a fixed point of the renormalization R, then $[\forall a>0,\ R(S_a\phi^*)=S_a\phi^*]$: either ϕ^* is totally symmetric, which means that $[\forall a>0, S_a\phi^*=\phi^*]$, either $\partial_a[S_a\phi^*](a=1)$ is a *marginal direction* of $DR(\phi^*)$ associated with the eigenvalue 1.

4 - Conclusion

Let me summarize the main assertions about the relative status of regularization and renormalization group.

• Regularization $\mathcal{R}_\infty(\Lambda)$ involves "*infinite cutoff*", replacing the diverging or doubtful contributions of momenta $q > \Lambda$ by *effective low-energy terms* ($q \leq \Lambda$). The idea is to replace a (more often infinite) set of virtual high-energy mechanisms, unphysical as no information is experimentally available on their role and on their properties, by their resulting finite effects at low energy, at scales where the theory is valid and experimental observations possible.

• Regularization is in fact *always present as a preliminary step*. It amounts either to a coarse-graining defining the minimum scale a in the real space, or to an ultra-violet cutoff $\Lambda = 2\pi/a$ in the conjugate space. Such a step has to be performed in order to obtain a workable model in which parameters can be fitted in using experimental observations.

• Each renormalization transformation $\mathcal{R}_k$ of a renormalization group $(\mathcal{R}_k)_{k>0}$ can be viewed as a "*finite regularization*": when performed in the conjugate space, $\mathcal{R}_k$ (or $\mathcal{R}_{k,\Lambda}$ in case of a non-autonomous renormalization group) amounts to "eliminate" the modes $\Lambda < q \leq k\Lambda$ (instead of the modes $q > \Lambda$).

• The various possible regularizations $[\mathcal{R}_\infty(\Lambda)]_{\Lambda>0}$, each associated with a cutoff parameter Λ, *are related through a renormalization group* $(\mathcal{R}_k)_{k>0}$; formally:

$$\mathcal{R}_\infty(\Lambda) = \mathcal{R}_k \circ \mathcal{R}_\infty(k\Lambda) \qquad \text{hence} \qquad \mathcal{R}_K = \mathcal{R}_k \circ \mathcal{R}_{K/k}$$

This renormalization group is more often non-autonomous:

$$\mathcal{R}_\infty(\Lambda) = \mathcal{R}_{k,\Lambda} \circ \mathcal{R}_\infty(k\Lambda) \qquad \text{hence} \qquad \mathcal{R}_{K,\Lambda} = \mathcal{R}_{k,\Lambda} \circ \mathcal{R}_{K/k,k\Lambda}$$

• The renormalization group analysis *focuses on the renormalization action in a space of model systems*, especially on fixed points and on asymptotic behavior of the renormalization flow; the basin of attraction of a fixed point appears as a *universality class*.

• In case of a *Lie group*, infinitesimal analysis is sufficient; that is, analysis can be performed *in the Lie algebra generated by the infinitesimal renormalization operators*.

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